On $\cos\sqrt{x}$

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1 Initial Thoughts

In this note, we will consider the function F, given by $F(x) = \cos\sqrt{x}$. This function exhibits some unexpected behavior; it was brought to my attention a dozen years or so ago by the late J. Jerry Uhl, who was a professor of mathematics at the University of Illinois.

According to the standard conventions of elementary calculus, the domain of F is the interval $[0,\infty)$. Moreover, an easy application of the chain rule—a standard differentiation procedure of elementary calculus—tells us that

$$F'(x) = -\frac{\sin\sqrt{x}}{2\sqrt{x}}. \tag{1}$$

So we conclude that the domain of F' is $(0,\infty)$, F itself not being defined to the left of the origin. (And, just to make the conclusion even more certain, there is a zero in the denominator at the origin in the expression on the right-hand side of equation (1).)

Thus, it comes as a surprise to many that substituting $\sqrt{x}$ for u in the Maclaurin series—that is, the Taylor series expanded about the origin

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots \tag{2}$$

gives us a Maclaurin series for a function φ

$$\varphi(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots \tag{3}$$

which agrees with F on $[0,\infty)$—but converges *for all real values of x.*

This phenomenon seems, at first glance, inconsistent with things we have learned about finding Taylor series. Among other things, the existence of all derivatives at the center of an expansion seems a prerequisite for existence of a series representation—not only, we recall, *at* the center, but *near* it. For example, a standard statement of Taylor's theorem with remainder—which is a basic tool for finding Taylor series—tells us that, for a non-negative integer n, and an arbitrary function f, possessing $n+1$ derivatives on an interval $(a-h, a+h)$ for some $h > 0$, we may write, for any $x \in (a-h, a+h)$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x), \tag{4}$$

where $R_n(x)$ is a remainder whose structure the theorem describes by making reference to $f^{(n+1)}$. This suggests that we need to be sure that f possesses at least $n+1$ derivatives at every point of some open interval centered at a. And, of course, if we're going to write a series expansion for f at $x = a$, this seems to require that we must be able to find, for f, derivatives of *all orders in some open interval centered at* $x = a$.

2 A Useful Digression

Before we think more about the function of the title of this essay, let's look at some more familiar, and more easily approached, examples.

Example 1 *The sinc function*

Consider the function s, defined by

$$s(x) = \frac{\sin x}{x}. \tag{5}$$

It's clear that the domain of s, as defined by equation (5), is $(-\infty, 0) \cup (0, \infty)$.

However, we learned, when we learned how to find the derivative of the sine function, that

$$\lim_{u \to 0} \frac{\sin u}{u} = 1. \tag{6}$$

Thus, if we define a new function S by

$$S(x) = \begin{cases} \dfrac{\sin x}{x}; & \text{when } x \neq 0 \\ \\ 1; & \text{when } x = 0, \end{cases} \tag{7}$$

we find that S is defined for all real numbers, and that S is continuous everywhere.

Moreover, if $x \neq 0$, we know that

$$S'(x) = \frac{x \cos x - \sin x}{x^2}, \tag{8}$$

while

$$\begin{aligned} S'(0) &= \lim_{h \to 0} \frac{S(0+h) - S(0)}{h} && (9) \\ &= \lim_{h \to 0} \frac{\frac{\sin h}{h} - 1}{h} && (10) \\ &= \lim_{h \to 0} \frac{\sin h - h}{h^2}. && (11) \end{aligned}$$

Applying l'Hôpital's rule (twice), we find that

$$\lim_{h \to 0} \frac{\sin h - h}{h^2} = \lim_{h \to 0} \frac{\cos h - 1}{2h} = \lim_{h \to 0} \frac{-\sin h}{2} = 0. \tag{12}$$

Thus, $S'(0) = 0$, and we see that S is differentiable everywhere.

What about S''?. That's easy when $x \neq 0$: for such x, we know that

$$S''(x) = \frac{2 \sin x - 2x \cos x - x^2 \sin x}{x^3}. \tag{13}$$

But to find $S''(0)$, we must again resort to the definition of the derivative:

$$\begin{aligned} S''(0) &= \lim_{h \to 0} \frac{S'(h) - S'(0)}{h} = \lim_{h \to 0} \frac{\frac{h \cos h - \sin h}{h^2} - 0}{h} && (14) \\ &= \lim_{h \to 0} \frac{h \cos h - \sin h}{h^3} = \lim_{h \to 0} \frac{-h \sin[h]}{3h^2} && (15) \\ &= -\frac{1}{3} \lim_{h \to 0} \frac{\sin h}{h} = -\frac{1}{3}. && (16) \end{aligned}$$

This means that S is twice differentiable throughout its entire domain.

Now it's easy to convince oneself that $S^{(n)}(x)$ exists for all n and for all $x \neq 0$; this follows from the standard differentiation rules. We could try to show that $S^{(n)}(0)$ exists for every positive integer n by continuing the line of arguments we have just begun as we demonstrated the existence of $S'(0)$ and $S''(0)$; if we succeeded, we would establish that S has derivatives of all orders at every real number.

But this program is one that would be hard, at best, to carry through. The derivatives $S^{(n)}(x)$ clearly become more and more complicated as n increases, and writing out a general formula promises to be difficult, indeed—if it is possible at all. Unless we can do so, it seems unlikely we can make the program work.

But, fortunately, we know some other things. For example,

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}, \tag{17}$$

whatever the real number x may be. Hence, when $x \neq 0$.

$$\frac{\sin x}{x} = \frac{\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}}{x}, \tag{18}$$

and, canceling the x in the denominator through all terms of the series, we find that

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!}. \tag{19}$$

Now we've discovered that the function s, in spite of its singularity at the origin, has a Maclaurin series representation, given by (19). After all, the series on the right side of equation (17) converges for *all* values of x.

But wait! The function s isn't even defined at $x = 0$, so how can it have a Maclaurin expansion? The situation is very similar to something we're already familiar with. We're used to saying things like, *As long as* $x \neq 0$*, we have*

$$\frac{x(x-2)}{x} = \frac{\cancel{x}(x-2)}{\cancel{x}} = x - 2, \tag{20}$$

and most of us don't worry about things like that any more.

The function S somehow naturally underlies s, just as $(x-2)$ underlies the fractions of equation (20); the Maclaurin expansion we've just found is really that of S. And we know, from what we've learned about power series, that because S is given by a power series that converges for all real numbers, it must have derivatives of all orders everywhere—including the origin.

The function S turns out to be an important function in certain areas of mathematics. In fact, it's important enough to have a widely accepted name: it's called *the sinc function,* and it's written $\operatorname{sinc} x$. It's usually taken to be defined by its Maclaurin series:

$$\operatorname{sinc} x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!}. \tag{21}$$

Note that taking equation (21) as the definition for this function not only avoids the inconvenient—in fact, *embarassing*—singularity at the origin of the function we've called s. It establishes the everywhere infinitely differentiable property of the sinc function as well.

Example 2 *The geometric series*

Let ψ be the function defined by the equation

$$\psi(x) = \sum_{k=0}^{\infty} x^k. \tag{22}$$

Yes, that's a simple geometric series on the right side, and we know that it converges precisely when $-1 < x < 1$. Moreover, we know what it converges to. So we know that an alternate description of ψ is that

$$\psi(x) = \frac{1}{1-x}, \text{ when } -1 < x < 1. \tag{23}$$

That innocent-looking compound inequality, "$-1 < x < 1$", is an essential part of this alternate description of ψ. Be sure you understand that ψ, because it's *defined* by equation (22), has just the interval $(-1, 1)$ for its domain, and we probably shouldn't expect that ψ has a Taylor expansion about $x = -1$, where it isn't even defined.. Once you're clear on this, we can proceed.

If $-1 < x < 1$, we can certainly write

$$\psi(x) = \frac{1}{1-x} = \frac{1}{1+(1-1)-x} \tag{24}$$

$$= \frac{1}{(1+1)-(x+1)} \tag{25}$$

$$= \frac{1}{2-(x+1)} \tag{26}$$

$$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}(x+1)}. \tag{27}$$

But we can rewrite the compound fraction on the right side of equation (27) as a geometric series,

$$\frac{1}{1-\frac{1}{2}(x+1)} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (x+1)^n, \tag{28}$$

provided only that

$$\left|\frac{1}{2}(x+1)\right| < 1, \text{or, equivalently,} \tag{29}$$

$$|x+1| < 2. \tag{30}$$

Thus, equation (28) converges on the interval $(-3, 1)$, and defines a new function Ψ, that agrees with ψ on the domain of the latter, and whose own domain extends the domain of the old function ψ. In this respect, what we see here is very similar to what we saw in the previous example with the sinc function and the function s. But there's a new wrinkle: the sinc function extended the function s, and its domain, by just one point, while the function Ψ and its domain extend quite a bit—over an interval, instead of a single point—to the left of the domain of ψ.

What consolidates our observations regarding the two functions ψ and Ψ is the knowledge that both of these functions have a mutual *global* extension: the function $x \mapsto (1-x)^{-1}$. That latter function is a "natural" extension of both ψ and Ψ, just as the function $x \mapsto x-2$ extends the function $x \mapsto (x^2-4)(x+2)^{-1}$ in a "natural" way. Moreover, we can't extend $x \mapsto (1-x)^{-1}$ any more in what we would think of as a "natural" way, the singularity at $x = 1$ being unfixable.

We can, immediately, take a couple of important lessons from these two examples:

Lesson 1 The fact that a function g doesn't seem to have higher order derivatives at $x = a$ doesn't preclude the existence of a Taylor expansion about $x = a$ for g. What it does do is *keep us from using Taylor's theorem with remainder to construct that Taylor expansion.* There are *other* ways to construct Taylor series, and if one of those ways succeeds, it demonstrates that those higher order derivatives do, indeed, exist (if not for the function as it was given, then at least for an extended version of the function).

Lesson 2 Some functions have extensions that are, in some sense, *natural.* We have examined some functions for which these extensions seem to arise as parts of "larger" functions—if only we can figure out what those larger functions are. (What have seen here, and what we are about to see, offer glimpses of a fascinating landscape—which is a major sub-topic, called *analytic continuation,* of the field of complex analysis.)

3 Back to The Heart of the Matter

Now let's take a closer look at $F'(x)$. (Recall: $F(x) = \cos\sqrt{x}$ for $0 \le x$.) Initially, we'll approach F as though we knew nothing about φ.

As we know, the function F is continuous on $[0, \infty)$. We know, too, that $F'(0)$ doesn't exist: after all, F isn't even defined to the left of $x = 0$. But what about $F'_+(0)$? Let's approach that question through the definition of the right-hand derivative:

$$F'_+(0) = \lim_{h \to 0^+} \frac{F(h) - F(0)}{h}. \tag{31}$$

Now, whatever $h > 0$ may be, F is surely continuous on the interval $[0, h]$ and differentiable on the interval $(0, h)$, so the Mean Value Theorem guarantees that there is a number $\xi_h \in [0, h]$ (so that $\xi_h \to 0^+$ as $h \to 0^+$) such that

$$F(h) - F(0) = F'(\xi_h)(h - 0) = F'(\xi_h)h. \tag{32}$$

Hence,

$$\lim_{h\to 0^+} \frac{F(h) - F(0)}{h} = \lim_{h\to 0^+} \frac{F'(\xi_h)\cancel{h}}{\cancel{h}} = \lim_{h\to 0^+} F'(\xi_h), \tag{33}$$

provided this latter limit exists.

But equation (1) gives F'. So

$$\lim_{h\to 0^+} F'(\xi_h) = \lim_{h\to 0^+} \left[-\frac{\sin\sqrt{\xi_h}}{2\sqrt{\xi_h}} \right] = -\frac{1}{2}. \tag{34}$$

because we know that $\dfrac{\sin t}{t} \to 1$ as $t \to 0$.

Remark With the argument we have just been through, we have, in passing, proved a theorem: *If a function f is continuous on $[a, b]$ and differentiable on (a, b), and if $\lim_{t\to a^+} f'(t) = L$, then $f'_+(a)$ exists and equals L.*

In fact, recalling Example 1, we now recognize that, when $0 < x < \infty$,

$$F'(x) = -\frac{1}{2} \operatorname{sinc} \sqrt{x}. \tag{35}$$

That's like a bucket of cold water in the face! The function F has domain $[0, \infty)$, but its derivative agrees, inside that domain, with a function we can extend, in a very natural way, to a function that has domain $\mathbb{R}$!

That's very interesting, because it gives us still another way to extend the definition of F—which we've already extended to a globally defined function φ by means of a Maclaurin series for F. Now we can write

$$\Phi(x) = 1 - \frac{1}{2} \int_0^x \operatorname{sinc} \sqrt{t}\, dt \tag{36}$$

to define a function Φ so that

1. Φ is globally defined: its domain is $\mathbb{R}$.
2. If $0 \le x < \infty$, then $\Phi(x) = F(x)$.
3. Φ has derivatives of all orders at every $x \in \mathbb{R}$.

The second item above is immediate from the observation that $F(0) = 1 = \Phi(0)$ and our earlier evaluation of $F'(0)$, while the third is an easy consequence of the Fundamental Theorem of Calculus and our earlier examination of the higher order derivatives of the sinc function.

There's just one little problem with equation (36): In order to carry through the promised extension, we must again resort to evaluating a series at imaginary values of its argument. After all, when $t < 0$, the expression $\operatorname{sinc}\sqrt{t}$ involves evaluation of the series in equation (21) at the imaginary number $x = i\sqrt{-t}$ where, as is customary, i stands for $\sqrt{-1}$ so that $i^2 = -1$. Of course, the series contains only even powers of x, so setting $x = i\sqrt{-t}$ results in a real outcome when we multiply things out and add them up—just as does the series we obtained for $\cos\sqrt{x}$ when x is negative. All of this raises another question we should deal with: what on earth can it mean to take the cosine of an imaginary number? Can we understand this except through its rather non-intuitive definition by means of a series whose terms require the use of imaginary numbers?

The answer to that question is that we can—if we are comfortable with the Euler relation:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{37}$$

(One way to be comfortable with this relation is to take it as the *definition* of the symbol $e^{i\theta}$. That is, we can simply agree that $\cos\theta + i\sin\theta$ is what $e^{i\theta}$ means, and that, with this understanding, the rules of algebra with exponents extend to complex exponents.)

If $e^{i\theta} = \cos\theta + i\sin\theta$, then $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$. Adding this latter equation to equation (37), we find that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, or

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{38}$$

Similarly, a subtraction leads to

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{39}$$

Now substitute iu for θ in the equation 38 to get:

$$\cos iu = \frac{e^{i\cdot iu} + e^{-i\cdot iu}}{2} = \frac{e^{i^2u} + e^{-i^2u}}{2} = \frac{e^{-u} + e^{-(-u)}}{2} \tag{40}$$

$$= \frac{e^{u} + e^{-u}}{2} = \cosh u. \tag{41}$$

Consequently, for those values of $x < 0$, it follows that $-x > 0$ and we have $\cos\sqrt{x} = \cos i\sqrt{-x} = \cosh\sqrt{-x}$. Therefore, we may write the function φ as

$$\varphi(x) = \begin{cases} \cos\sqrt{x}, & \text{when } x \geq 0; \\ \\ \cosh\sqrt{-x}, & \text{when } x < 0. \end{cases} \tag{42}$$

We remark that it is easy to check, using the Maclaurin expansion

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \cdots\,, \tag{43}$$

that equations (3), (36), and (42) really do give exactly the same function. See Figure 1.

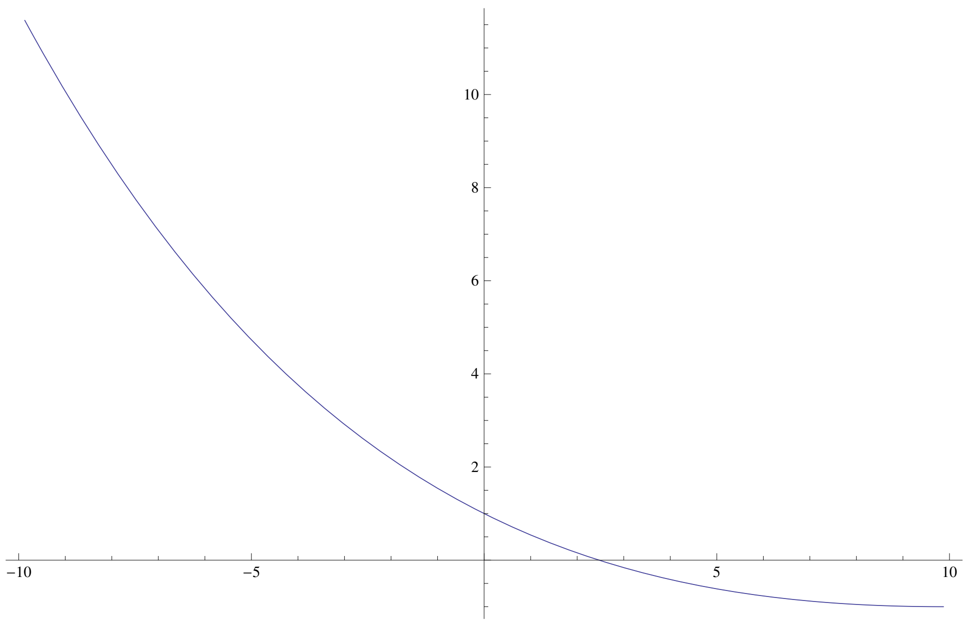

Figure 1: $y = \cos\sqrt{x}$

It may be tempting, on account of equation (42), to call φ a "piecewise" function—but, because of equation (3), and, especially, in the light of the way that the function φ seems, somehow, to be more fundamental than the function F, that's a misnomer.

Indeed, the function φ can be thought "piecewise" only if we refuse to allow complex numbers into our discussions. When we allow them, the equation $\varphi(x) = \cos\sqrt{x}$ gives a perfectly good description of our function. Refusal to consider complex numbers is, unfortunately, an intransigence we seem to be stuck with in the elementary calculus sequence—in spite of our insistence on including them in our algebra curriculum. And, even when we refuse to admit that there are such things as complex numbers, it is still easy to describe φ through a power series.

Thus, we can take a final lesson from the example:

Lesson 3 The notion of a *piecewise function* is an artificial construct, arising from a misplaced desire to describe the underlying natures of particular functions in terms of certain of their representations. It is our *representations* that are piecewise—not the *functions* themselves.