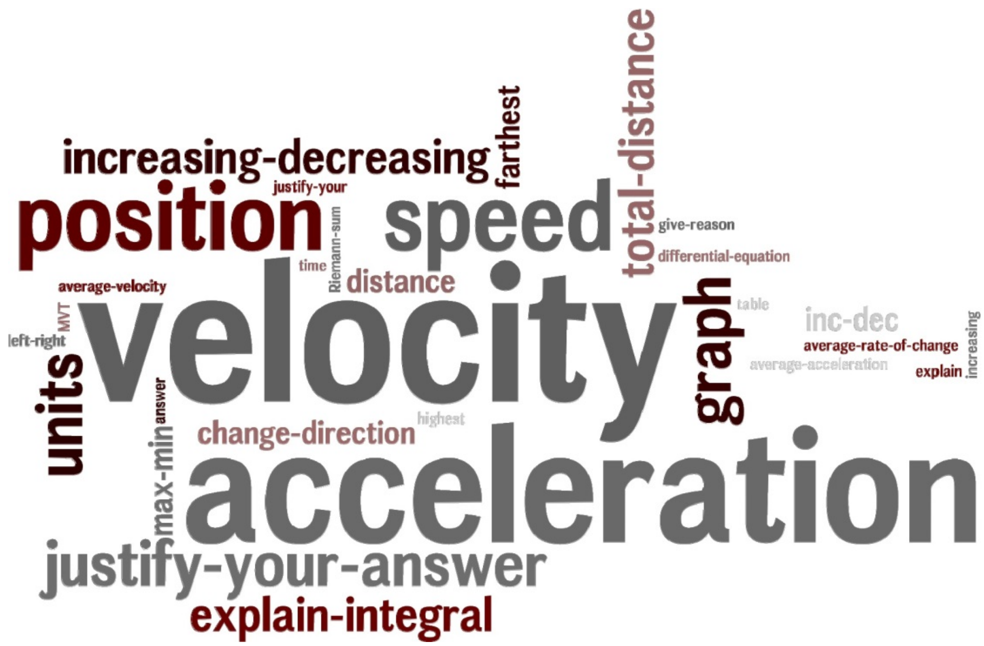

The Ubiquitous Particle Motion Problem

Presented by Lin McMullin
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"A particle is moving along the *x*-axis …."

So begins a number of AP Calculus questions. Sometimes it's a particle, sometimes a car, or a rocket. Once it was a pair of former calculus chief readers.

Usually the velocity is given and students are asked questions about the position, the acceleration, the speed, or the direction of motion. Sometimes the motion is described by an equation, sometimes a graph and occasionally by a table.

While it may take some time to actually find a particle that moves according to the functions used on the AP exams, the format allows students to be tested on a great variety of important calculus concepts: limits, derivatives, integrals, accumulation, and initial value differential equations problems. This is why motion problems appear so often on the exams. But, you will find very few of these rich problems in textbooks. Textbooks often consider only uniformly accelerated motion – a grapefruit thrown up into the air. And that, in turn, is why AP calculus teachers must supplement their textbooks on this topic.

The Basics

Position

- The *position* of an object *x*(*t*) is a function of time, *t*. The object moves on a line toward or away from a fixed reference point (e.g. the origin).

Velocity

- The *velocity* of the object is the derivative of the position: $v(t) = x'(t)$.
- Velocity has direction as indicated by its sign and a magnitude called *speed*. Thus, velocity is a vector. (The term "vector" will not appear on the AB exam.)
- Positive velocity indicates motion in the positive direction; negative velocity indicates motion in the negative direction.
- The units of velocity are distance divided by time (miles/hour, feet per second, etc.).
- Velocity is the antiderivative of acceleration.
- Position is the antiderivative of velocity.
- You may think of the velocity as pulling the particle one way or the other.

Acceleration

- The *acceleration* of the object is the derivative of the velocity and the second derivative of the position: $a(t) = v'(t) = x''(t)$.
- Acceleration is also a vector. Its sign indicates the direction in which the velocity is changing: positive acceleration indicates the velocity is increasing; negative decreasing.
- The units of acceleration are velocity divided by time ((miles/hour)/hour, (feet per second) per second, etc.).

Speed

- The *speed* of the object is the absolute value of the velocity: $\text{Speed } = |v(t)|$.
- Speed is the length of the velocity vector.
- Speed is a number, not a vector.
- Its units are the same as velocity units (miles/hour, feet per second, etc.).
- Speed is *increasing* when the velocity and acceleration act in the same direction – they have the same sign; speed is *decreasing* when the velocity and acceleration act in different directions – they have different signs.
- If the velocity graph is moving *away from* the *t*-axis the speed is *increasing*; if the velocity graph is moving *towards* the *t*-axis the speed is *decreasing*.

Distance

- The total distance traveled over the time interval [*a, b*] is the definite integral of speed (think: rate times time is distance): $\int_a^b |v(t)|\,dt$
- The net distance traveled (displacement) over the time interval [*a, b*] is the definite integral of velocity: $\int_a^b v(t)\,dt = x(b) - x(a)$
- The position at any time *t* is the initial position, $x(a)$, plus the displacement: $x(t) = x(a) + \int_a^t v(T)\,dT$

Corresponding Concepts

A vocabulary exercise: Working around all these terms is the same "calculus" as appears in other equation, graph, and table problem situations. The chart below shows the correspondence between the terms used to describe functions and the terms used to describe linear motion.

Function	Linear Motion
Value of a function at *x*	Position at time *t*
First derivative	Velocity
Second derivative	Acceleration
Increasing	Moving to the right or up
Decreasing	Moving to the left or down
Absolute Maximum	Farthest right or up
Absolute Minimum	Farthest left or down
y' = 0.	"At rest"
y' changes sign	Object changes direction
	Speed
y' positive and increasing	Speed is increasing
y' negative and increasing	Speed is decreasing
y' positive and decreasing	Speed is decreasing
y' negative and decreasing	Speed is increasing

If the students are asked to find the time interval during which the particle is moving to the right, he uses the same techniques as when finding when a function is increasing: find where the velocity (first derivative) is positive. Or if asked for the farthest the particle moves to the left she should look for the absolute minimum value of the position by considering the critical values and the endpoints.

Some Examples from Past AB Exams

See the attached scoring samples.

Study past questions to learn how the questions are worded, the variety of topics that are woven into them and what is expected by readers. Multiple-choice questions test the same topics in smaller pieces.

2011 AB 1:

The stem gives the equations of the velocity and acceleration. This is a calculator active question so students should begin by putting both these equations in their calculator so they can recall them as needed. Notice that units were not required in this question.

(a) The first part asks whether the speed is increasing. Thus students have to find the sign of the velocity and acceleration and use them to determine the answer. They then must white a sentence telling how they know. Something like "The speed is decreasing at $t = 5.5$ because the velocity and acceleration have different signs," will suffice.
(b) Next an integration problems: find the average (value of the) velocity. An integration question.
(c) Next find the total distance. Another integration question.
(d) Finally, the question asks when and where the particle changes direction. So students have to find (1) where the velocity is zero by solve an equation on their calculator. Then (2) they need to find the position by integrating the velocity and adding on the initial position.

2006 AB 4:

Here the velocity of a rocket at various times is given in a table. In two of the parts units are required.

(a) The average acceleration is the average rate of change of velocity (derivative of velocity, slope of velocity).
(b) Students must explain the meaning of a definite integral in context which includes units. Then they have to approximate this integral using midpoint Riemann sum. The left- or right-sum or a trapezoidal approximation could just as well have been required.

(c) Here we switch to another rocket whose acceleration is given as an equation and students were asked which rocket is traveling faster after 80 second. Thus students must integrate the acceleration (by hand), add the initial velocity and compare the answer with the rocket in the table.

2009 AB 1:

This year the velocity of Caren riding to school on a bicycle was given as a graph. Again units were required in the first two parts of the problem.

(a) From the velocity graph students were asked to find the acceleration at a particular time. Acceleration is the derivative (slope) of velocity. Students read this from the graph. Units were required.
(b) Students were asked for the meaning of a definite integral of the speed and to find its value by working with the areas of the regions between the graph and the axis (integration). Again, units were required.
(c) In this part students needed to find when Caren turned around and returned home, that is they needed to find the location of the first local maximum.
(d) Finally, the velocity of Larry riding to school on his bicycle was given as an equation. Students needed to integrate this find his distance. Then they needed to find Caren's distance from school by finding the area of the appropriate region on the graph and comparing this to Larry's distance.

2008 AB 4:

Here again students faced a graph in the stem of the problem. The areas of the regions between the graph and the t-axis were given.

(a) Students were asked for the time when the particle was farthest to the left (i.e. find the absolute minimum) and its position. This occurs when the velocity (derivative) changes from negative to positive or at an endpoint. To justify the answer students had to compute both positions using the "position at any time" equation given above. The displacements were found from the graph and the given information.
(b) To find how many times the particle was at $x = -8$, students needed to calculate the position at the critical points and end points and use the intermediate value theorem to explain their reasoning.
(c) Another question about whether the speed is increasing or decreasing with a justification based this time on reading the graph.
(d) Finally, students were asked when the acceleration was negative and to justify their answer. The acceleration is negative when the velocity is decreasing.

2008 AB Multiple-choice questions:

Multiple-choice questions about particle motion situations also appear on the exams. They ask the same kinds of things in smaller bites. They also start with graphs and tables as well as equations. See for examples 2008 AB – 7, 21, 82, 86, 87 and 2012 AB – 6, 16, 28, 79, 83, 89.

As you can see the particle motion questions are great for asking all kinds of questions on all kinds of topics in different way. Yet, they are really just testing the same things that are tested in other contexts with different sounding questions that are really not that different at all.

With 5 or 6 multiple-choice questions and one free-response question motion problems make up about 14% of the exam.

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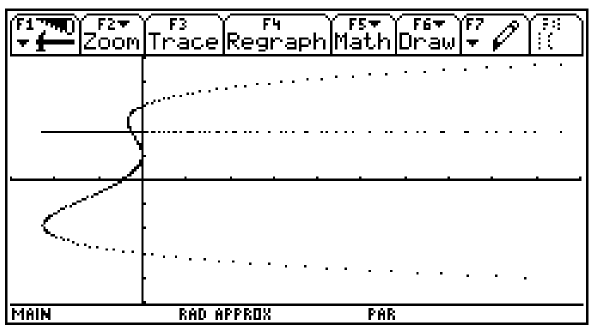
Extra

Here's how to show particle motion on your calculator:



1. Use these setting on your graphing calculator: In the parametric graphing mode, set you calculator to draw "Dot," and use these Window settings:

    tmin = –4, tmax = 5, tscl = 0.05,

    xmin = –3, xmax = 10, sscl = 1,

    ymin = –5, yamx = 5 and yscl = 1.

Enter this parametric equation: $xt1 = 0.05(t - 1)^2 (t^2 - 9)$ and $yt1 = 2$

2. Explorations:

    a. 1. GRAPH: What do you observe? Re-graph and watch again.

    b. 2. TRACE: Using the right arrow key you can move step by step along the graph starting from t = 0. Remember each push of the arrow moves you the same time unit along the path. What do you notice about the size and direction of the movement (look especially at those points to the left of the *y*-axis)? What does the distance between successive points tell you? Another thing you may try is to return to WINDOW and change t-step to a larger number, try 0.1 to speed things up.

3. Look from a different perspective: Return to the equation editor and make $xt2 = 0.05(t - 1)^2 (t^2 - 9)$ and $yt2 = t$. This will show you the particle's motion with one pixel in each row of pixels. Previously we were looking at the same particle from an edge-on view of the plane; now we are looking down on a plane in which the particle is moving.
4. Look at them both together: Draw both graphs on the same screen and compare them.

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For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

(a) $v(5.5) = -0.45337,\ a(5.5) = -1.35851$

The speed is increasing at time $t = 5.5$, since velocity and acceleration have the same sign.

2 : conclusion with reason

(b) Average velocity $= \frac{1}{6}\int_0^6 v(t)\,dt = 1.949$

2 : { 1 : integral; 1 : answer }

(c) Distance $= \int_0^6 |v(t)|\,dt = 12.573$

2 : { 1 : integral; 1 : answer }

(d) $v(t) = 0$ when $t = 5.19552$. Let $b = 5.19552$.

$v(t)$ changes sign from positive to negative at time $t = b$.

$x(b) = 2 + \int_0^b v(t)\,dt = 14.134$ or 14.135

3 : { 1 : considers $v(t) = 0$; 1 : integral; 1 : answer }

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

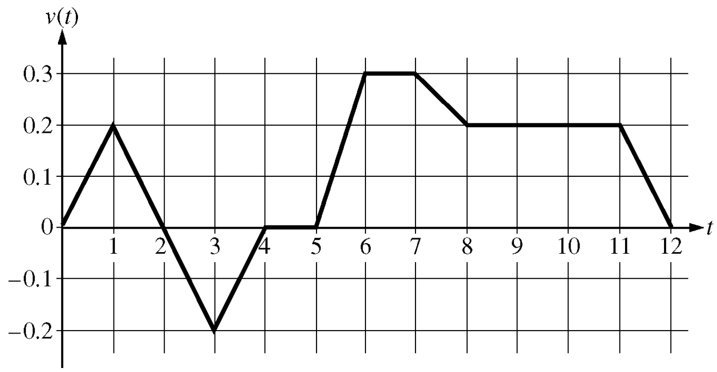
(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

Solution	Scoring
(a) Average acceleration of rocket A is $\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20}\ \text{ft}/\text{sec}^2$	1 : answer
(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20) + v(40) + v(60)]$ $= 20[22 + 35 + 44] = 2020$ ft	3 : { 1 : explanation 1 : uses $v(20)$, $v(40)$, $v(60)$ 1 : value }
(c) Let $v_B(t)$ be the velocity of rocket B at time t. $v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$ $2 = v_B(0) = 6 + C$ $v_B(t) = 6\sqrt{t+1} - 4$ $v_B(80) = 50 > 49 = v(80)$ Rocket B is traveling faster at time $t = 80$ seconds.	4 : { 1 : $6\sqrt{t+1}$ 1 : constant of integration 1 : uses initial condition 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion }
Units of ft/sec^2 in (a) and ft in (b)	1 : units in (a) and (b)

Question 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

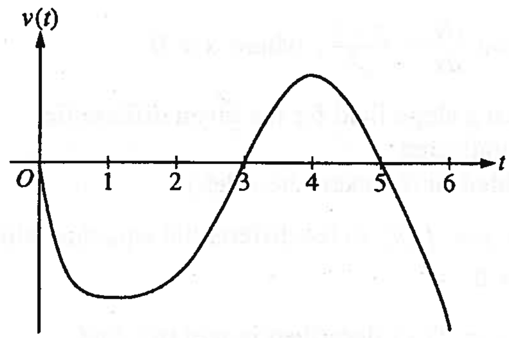
(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1 \text{ miles/minute}^2$	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{units} \end{cases}$
(b) $\int_0^{12} |v(t)|\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$. $\int_0^{12} |v(t)|\, dt = \int_0^{2} v(t)\, dt - \int_2^{4} v(t)\, dt + \int_4^{12} v(t)\, dt$ $= 0.2 + 0.2 + 1.4 = 1.8$ miles	$2 : \begin{cases} 1 : \text{meaning of integral} \\ 1 : \text{value of integral} \end{cases}$
(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school. $\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school. Therefore, Caren lives closer to school.	$3 : \begin{cases} 2 : \text{Larry's distance from school} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 1 : \text{Caren's distance from school} \\ \quad \text{and conclusion} \end{cases}$



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.

$$x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$$

$$x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$$

Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.

3 : { 1 : identifies $t = 3$ as a candidate; 1 : considers $\int_0^6 v(t)\,dt$; 1 : conclusion }

(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.

By the Intermediate Value Theorem, there are 3 values of t for which the particle is at $x(t) = -8$.

3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$; 1 : description of motion; 1 : conclusion }

(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.

1 : answer with reason

(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

2 : { 1 : answer; 1 : justification }

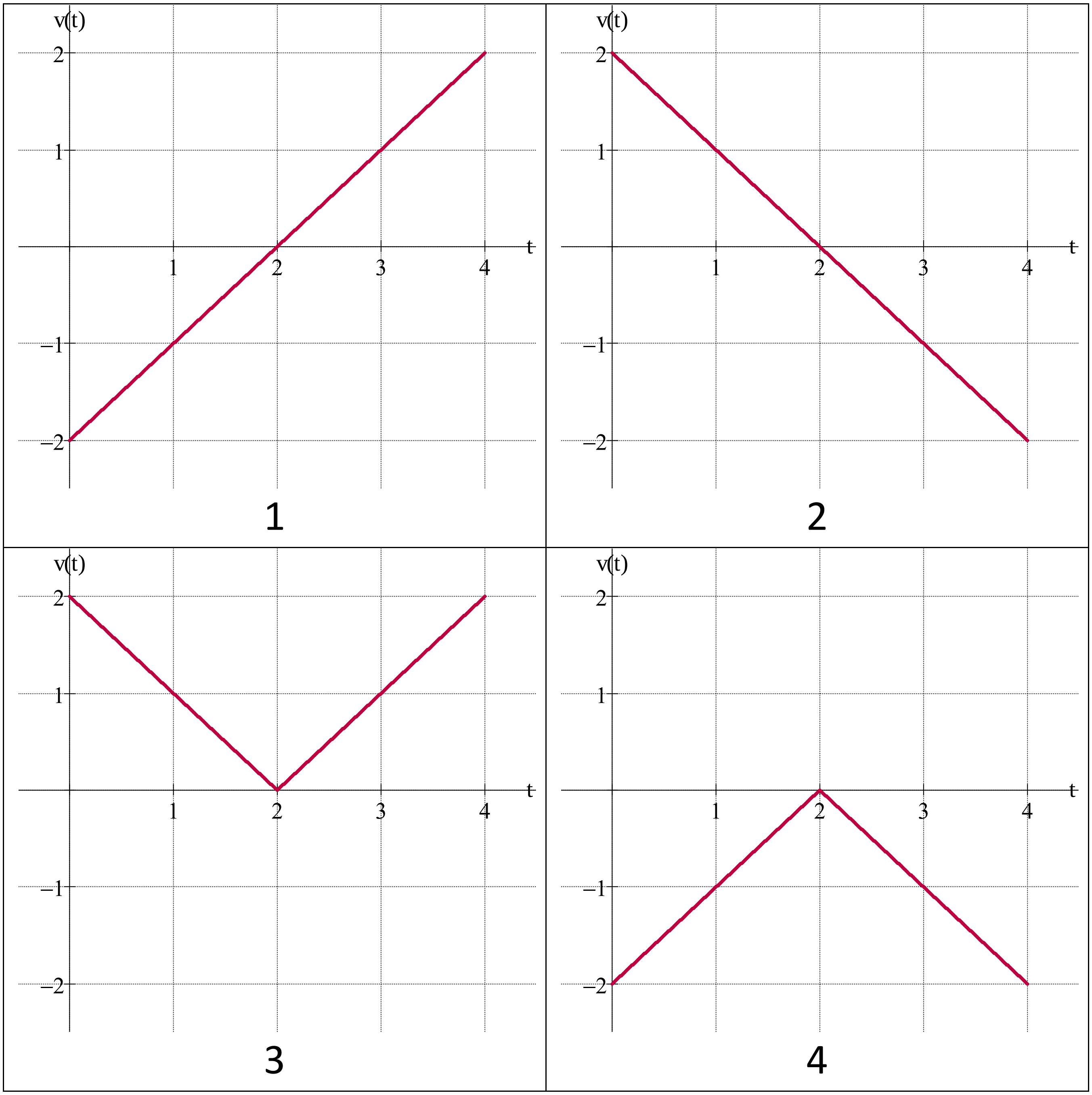
Velocity Matching Game

Name ______________________________

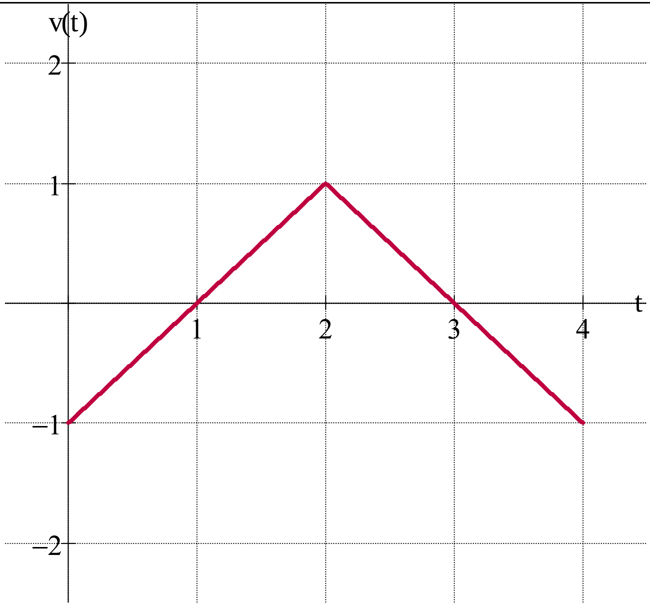
Answer Sheet.

Directions: Each of the cards handed out represents a particle moving (in feet) during its first 4 seconds. Each particle starts out at a position of 0 and begins moving as instructed on the card. The cards are divided into three categories: the velocity graph, a verbal description, and a position graph of the object in motion. Your goal is to match the cards and fill out your answers in the table provided below. Each Velocity Graph may have more than one Verbal Description and each Verbal Description my go with more than one Velocity Graph – Be sure you get them all. :

VELOCITY GRAPH (1-12)	VERBAL DESCRIPTION (A-L)	POSITION GRAPH (I-XII)
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		



v(t)
2
1
−1
−2
t
1
2
3
4
1
2
3
4



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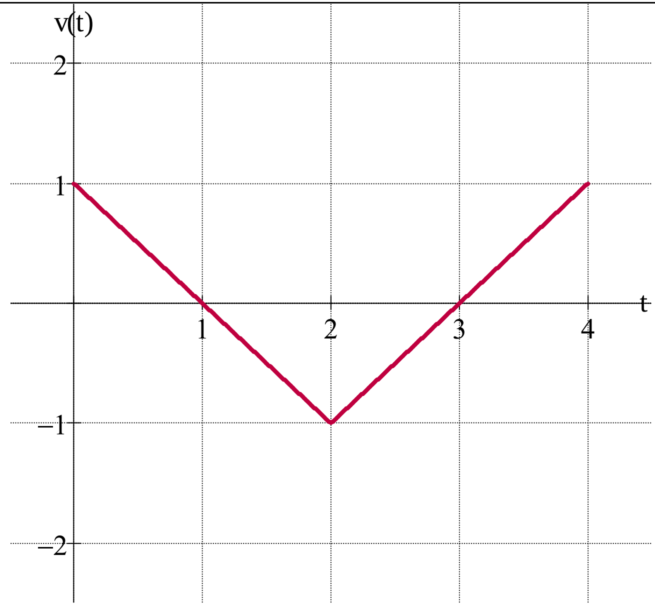

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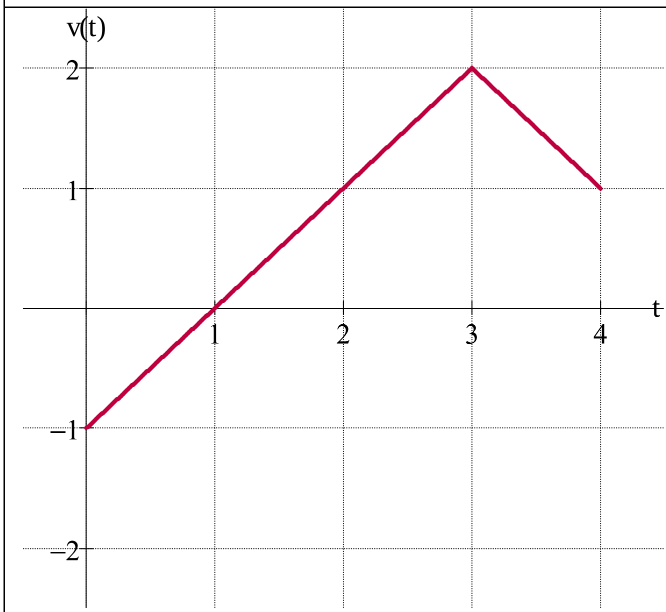

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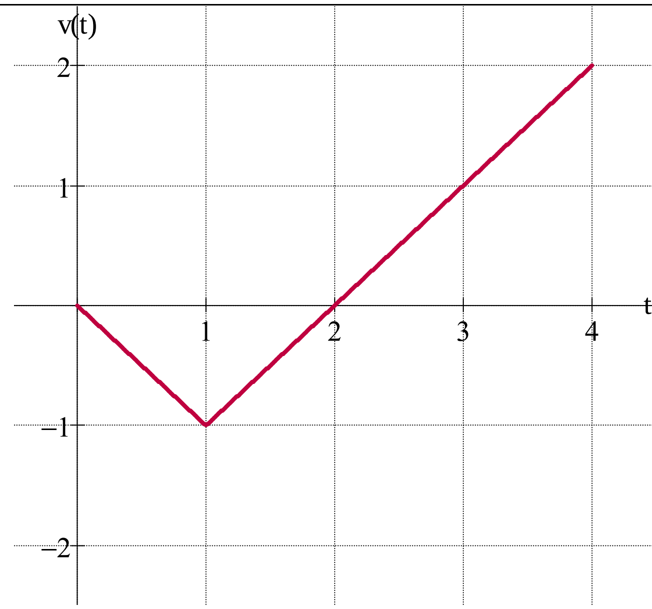

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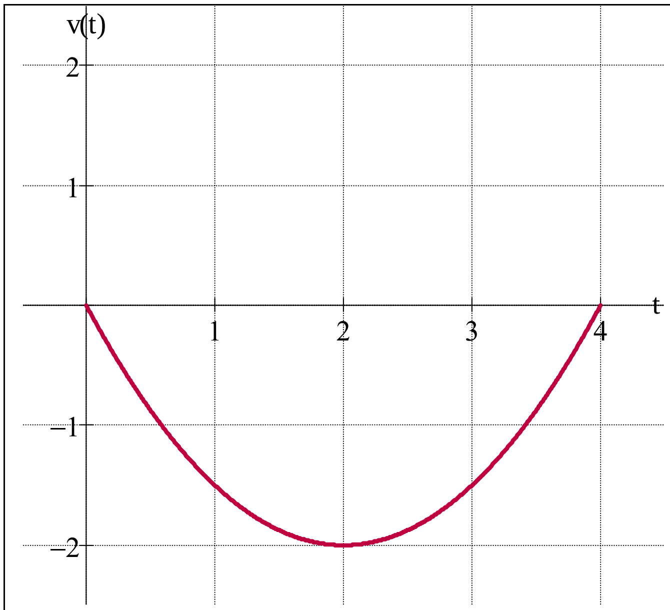

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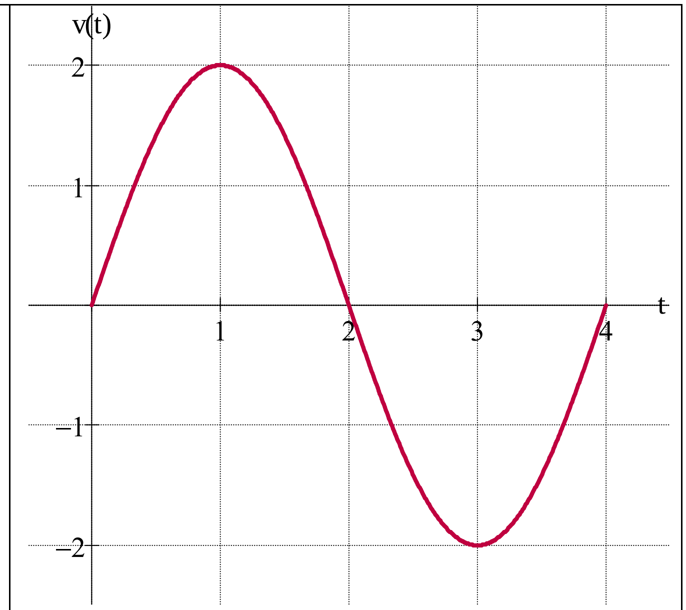

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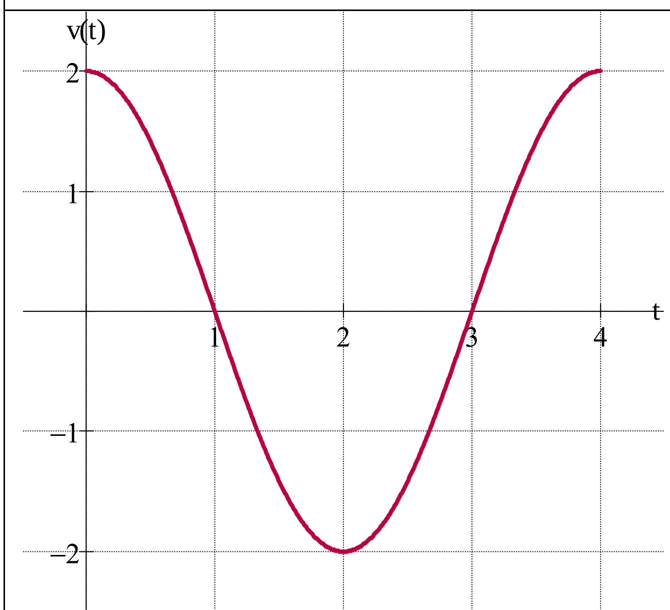

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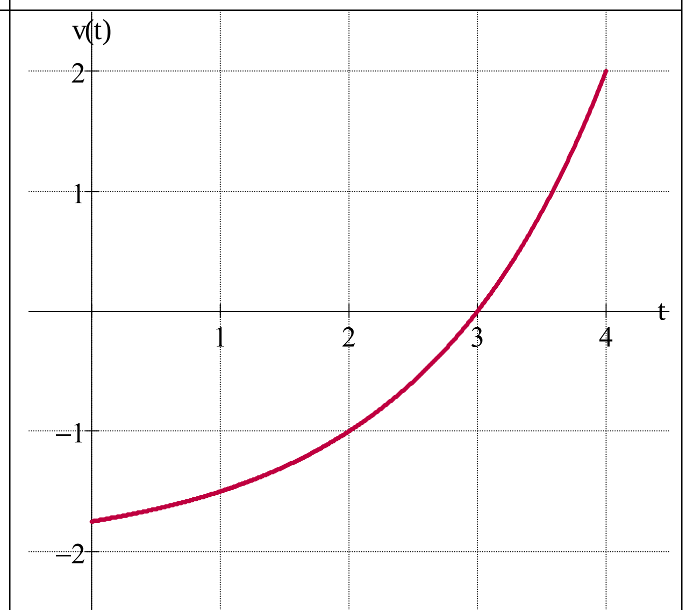

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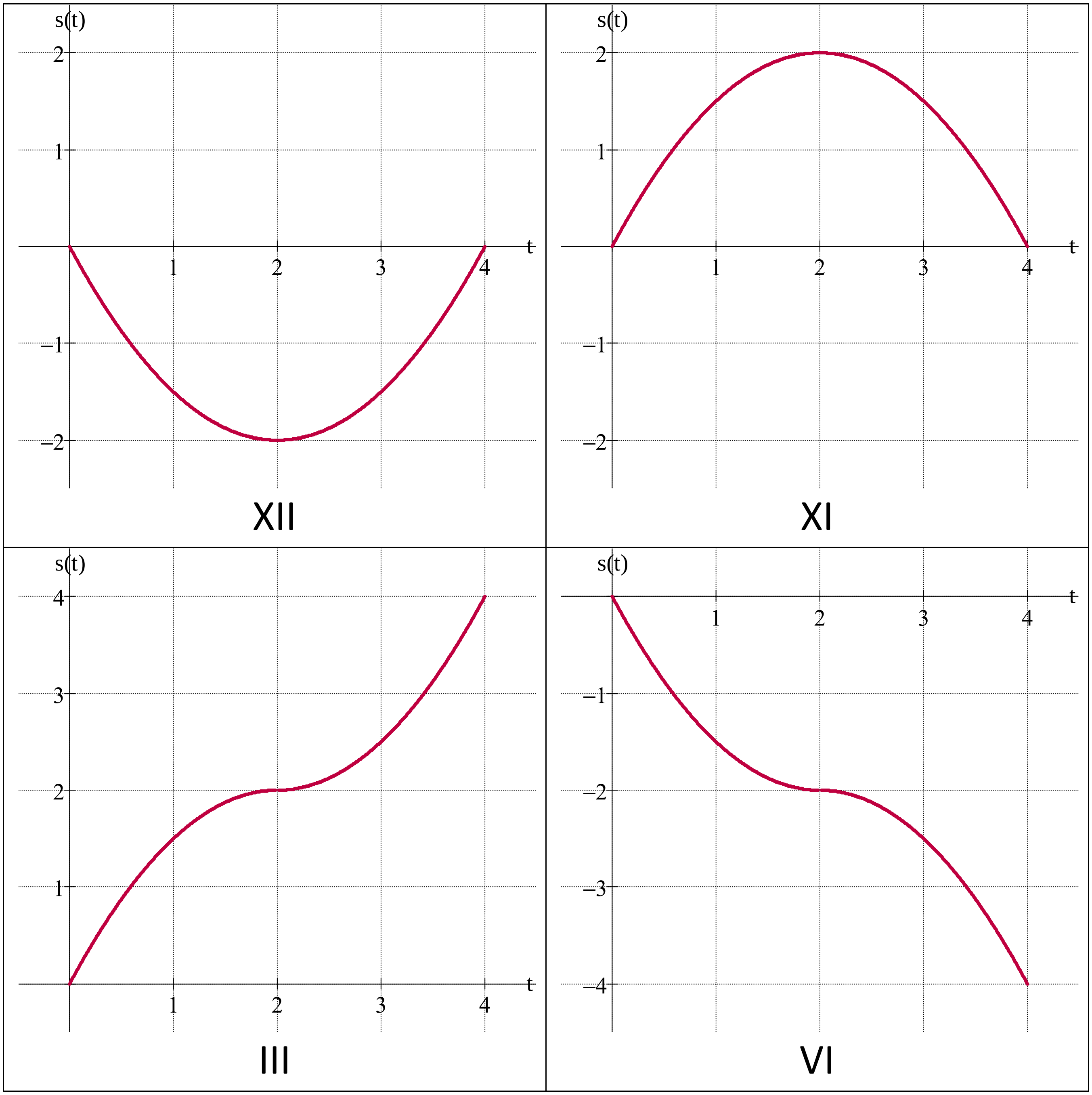
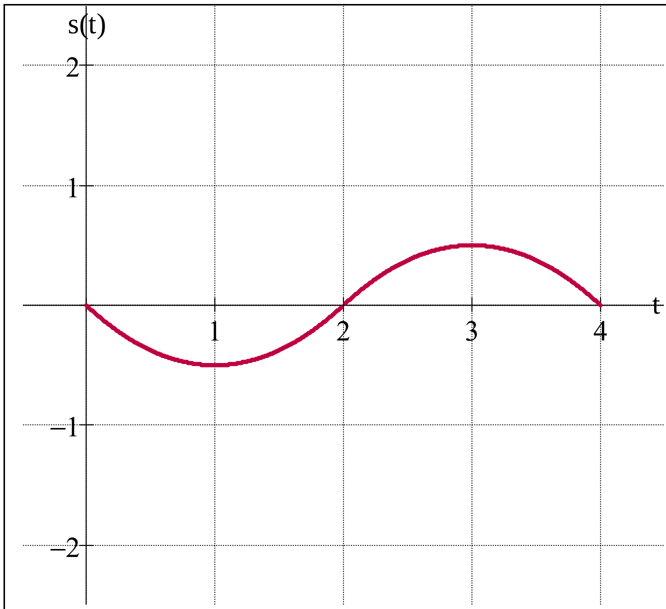
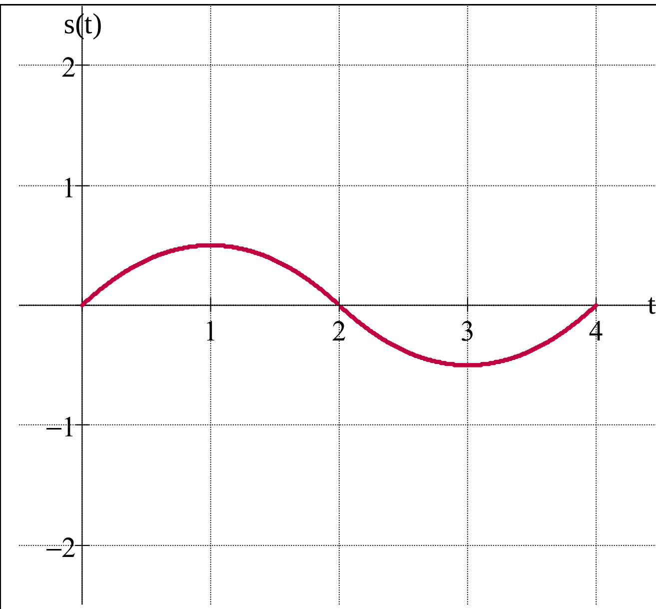
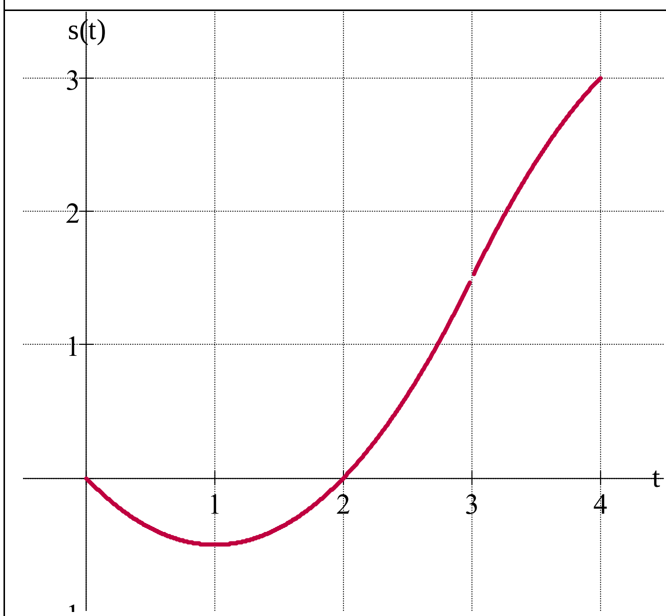

s(t)
t
XII
XI
III
VI



VIII



X



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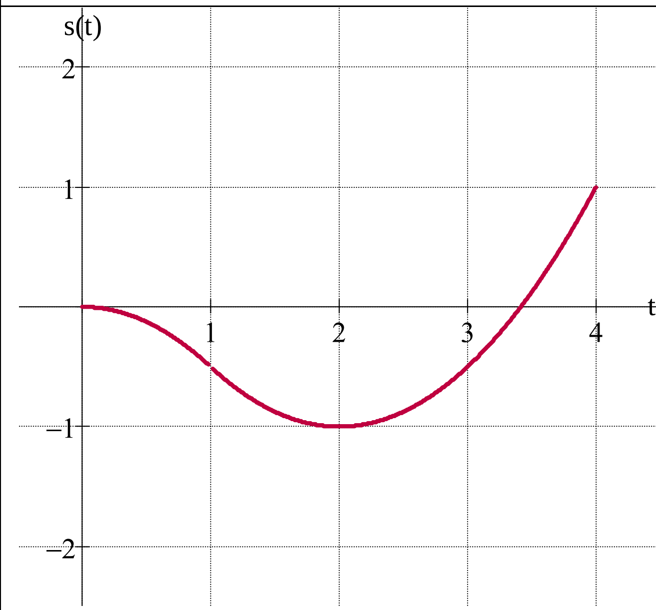
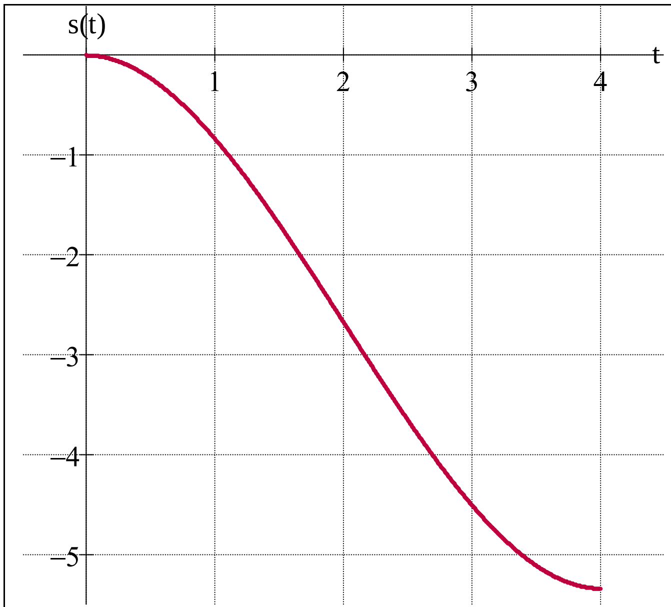


I



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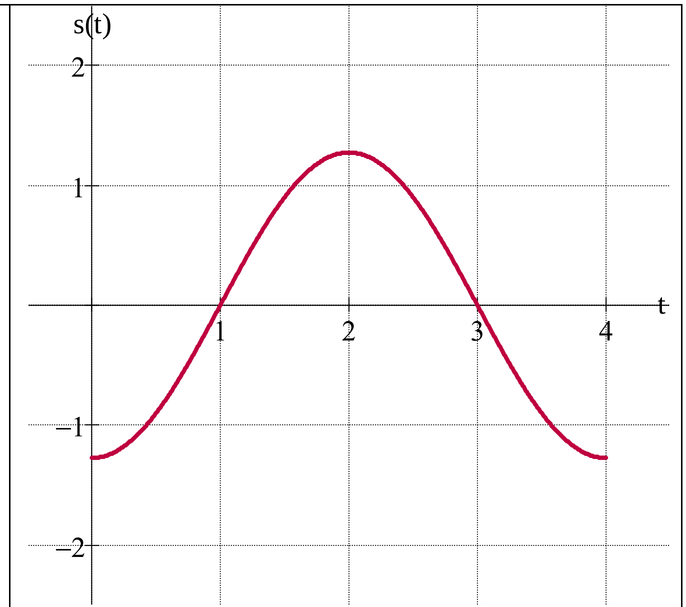

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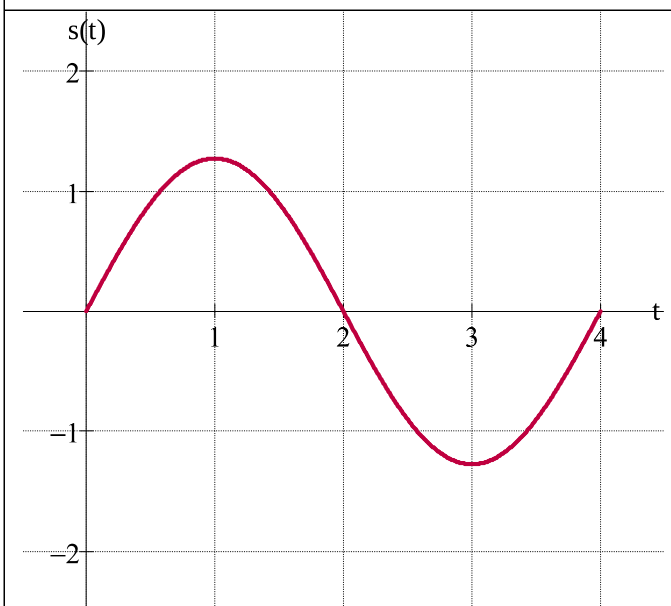

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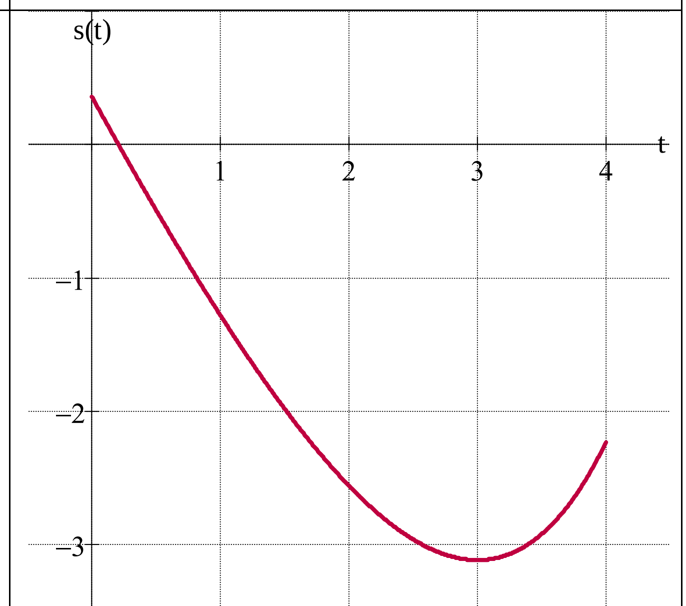


V

This particle changes direction at 2 seconds. A	This particle travels a net distance of 0 over the four seconds. K
This particle travels to the right the entire four seconds. C	This particle slows down and then speeds up. L

This particle stops at 2 seconds. D	This particle travels a net distance of three feet over the four seconds. G
This particle moves right and then left to end up back where it began. B	This particle travels farthest to the left overall. I

This particle changes direction twice over the four seconds. E	This particle speeds up, slows down, then speeds up and slows down again. F
This particle travels a total distance of three feet. H	This particle speeds up and then slows down. J

Answers

VELOCITY GRAPH (1-12)	VERBAL DESCRIPTION (A-L)	POSITION GRAPH (I-XII)
1	A, D, K, L	XII
2	A, B, D, K, L	XI
3	C, D, L	III
4	D, L	VI
5	D, E, K	VIII
6	E, G, K	X
7	B, G	IX
8	A, D, H, I	I
9	I, J	VII
10	A, B, D, F, K	II
11	E, H, K	IV
12	L	V

The Ubiquitous Particle Motion Problem

Presented by

Lin McMullin

NCTM Annual Meeting 2013

A Quick Look at Some Questions

2011 AB 1

For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

A Quick Look at Some Questions

2006 AB 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

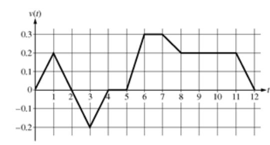
(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

A Quick Look at Some Questions

2009 AB 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\,dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\,dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

Velocity

- The velocity is the derivative of the position,

$$x'(t) = v(t)$$

- Velocity is has direction (indicated by its sign) and magnitude. Technically, velocity is a vector; the term "vector" will not appear on the AB exam.
- Velocity is the antiderivative of the acceleration
- Position is the antiderivative of velocity.

Acceleration

- Acceleration is the derivative of velocity and the second derivative of position,

$$a(t) = v'(t) = x''(t)$$

- It has direction and magnitude and is a vector.
- Velocity is the antiderivative of the acceleration

Speed

- Speed is the absolute value of velocity

$$\text{Linear motion's speed } = |v(t)| = \sqrt{\left(\tfrac{dx}{dt}\right)^2}$$

$$\text{In a plane, speed } = |\vec{v}(t)| = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}$$

- Speed is the length of the velocity vector
- Speed is a number, not a vector.

Speed

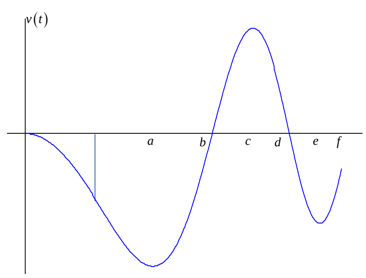
- How to determine if the speed is increasing or decreasing:

  ✓ *If the velocity and acceleration have the same sign, then the speed is increasing.*

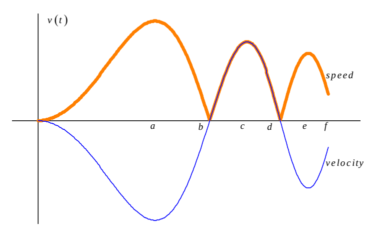
  ✓ *If velocity and acceleration have different signs, the speed is decreasing.*

  ✓ *If the velocity graph is moving away from (towards) the t-axis the speed is increasing (decreasing).*

Speed



Speed



Distance

- The total distance traveled is the definite integral of the speed

$$\int_a^b |v(t)|\,dt$$

- The net distance traveled (displacement) is the definite integral of the velocity (rate of change)

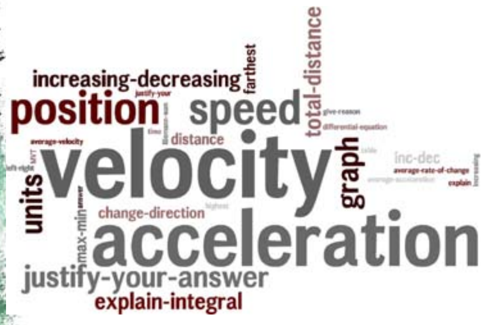
$$\int_a^b v(t)\,dt = x(b) - x(a)$$

Position

- The final position is the initial position plus the net distance traveled from *x* = *a* to *x* = *t*:

$$x(t) = x(a) + \int_a^t v(T)\,dT$$

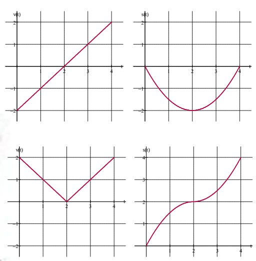
- Notice that this is an accumulation function equation.



Corresponding Concepts

Function	Linear Motion
Value of a function at *x*	Position at time *t*
First derivative	Velocity
Second derivative	Acceleration
Increasing	Moving to the right or up
Decreasing	Moving to the left or down
Absolute Maximum	Farthest right or up
Absolute Minimum	Farthest left or down
$y' = 0$.	"At rest"
y' changes sign	Object changes direction
	Speed
y' positive and increasing	Speed is increasing
y' negative and increasing	Speed is decreasing
y' positive and decreasing	Speed is decreasing
y' negative and decreasing	Speed is increasing

Velocity Matching Game



A. Changes direction at 2 seconds

D. Stops at 2 seconds

K. This particle travels a net distance of 0 over the four seconds.

L. This particle slows down and then speeds up.

C. This particle travels to the right the entire four seconds.

D. Stops at 2 seconds

L. This particle slows down and then speeds up.

By Brian Leonard,
Lake Hamilton (AR) High School

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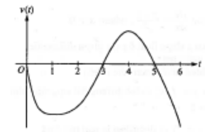
A Quick Look at Some Questions

2008 AB 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

Multiple-choice from AB 2008

7. A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \ge 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

(A) 4 (B) 6 (C) 9 (D) 11 (E) 12

87. An object traveling in a straight line has position $x(t)$ at time t. If the initial position is $x(0) = 2$ and the velocity of the object is $v(t) = \sqrt[3]{1 + t^2}$, what is the position of the object at time $t = 3$?

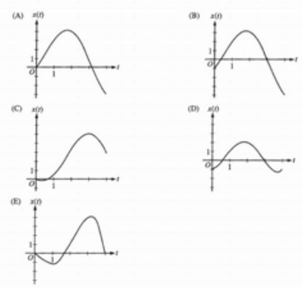
(A) 0.431 (B) 2.154 (C) 4.512 (D) 6.512 (E) 17.408

82. A particle moves along a straight line with velocity given by $v(t) = 7 - (1.01)^{-t^2}$ at time $t \ge 0$. What is the acceleration of the particle at time $t = 3$?

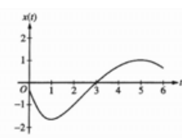
(A) −0.914 (B) 0.055 (C) 5.486 (D) 6.086 (E) 18.087

t	0	1	2	3	4
$v(t)$	−1	2	3	0	−4

86. The table gives selected values of the velocity, $v(t)$, of a particle moving along the x-axis. At time $t = 0$, the particle is at the origin. Which of the following could be the graph of the position, $x(t)$, of the particle for $0 \le t \le 4$?



Multiple-choice from AB 2008



21. A particle moves along a straight line. The graph of the particle's position $x(t)$ at time t is shown above for $0 < t < 6$. The graph has horizontal tangents at $t = 1$ and $t = 5$ and a point of inflection at $t = 2$. For what values of t is the velocity of the particle increasing?

(A) $0 < t < 2$

(B) $1 < t < 5$

(C) $2 < t < 6$

(D) $3 < t < 5$ only

(E) $1 < t < 2$ and $5 < t < 6$

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