Teaching Power Series
By Lin McMullin

From the BC Calculus Course Description:

***IV. Polynomial Approximations and Series**

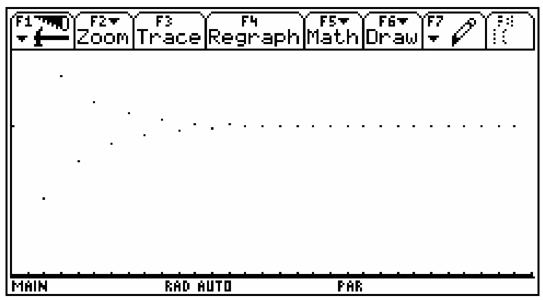
*** Concept of series.** A series is defined as a sequence of partial sums, and convergence is defined in terms of the limit of the sequence of partial sums. Technology can be used to explore convergence or divergence.

*** Series of constants.**
+ Motivating examples, including decimal expansion.
+ Geometric series with applications.
+ The harmonic series.
+ Alternating series with error bound.
+ Terms of series as areas of rectangles and their relationship to improper integrals, including the integral test and its use in testing the convergence of *p*-series.
+ The ratio test for convergence and divergence.
+ Comparing series to test for convergence or divergence.

*** Taylor series.**
+ Taylor polynomial approximation with graphical demonstration of convergence. (For example, viewing graphs of various Taylor polynomials of the sine function approximating the sine curve.)
+ Maclaurin series and the general Taylor series centered at $x = a$.
+ Maclaurin series for the functions , $\sin x$, $\cos x$, and $\dfrac{1}{1-x}$.
+ Formal manipulation of Taylor series and shortcuts to computing Taylor series, including substitution, differentiation, antidifferentiation, and the formation of new series from known series.
+ Functions defined by power series.
+ Radius and interval of convergence of power series.
+ Lagrange error bound for Taylor polynomials.

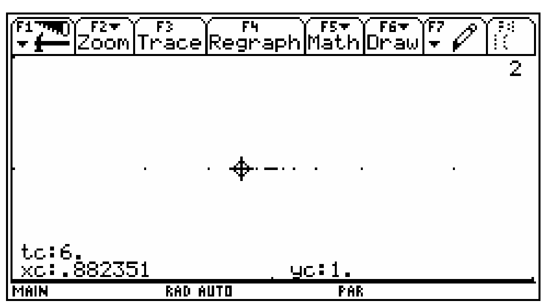
Investigation 1: Graphing sequences on a graphing calculator.

Most graphing calculators have built-in menus for graphing sequences of numbers. The technique described here may also be used. It makes use of the properties of sequences to produce the graph. A sequence is a function whose domain is the set of positive integers. On any calculator, graph in parametric form and dot mode (so the points will not be connected). You can see the individual terms of the sequence and see them converge. To graph the sequence $a_n = 1-(-0.7)^n$ in a plane enter $xt1(t) = t$ and $yt1(t)$ = 1– (–0.7)^t and use these window settings: tmin = 1, tmax = 30, tstep = 1, (this makes the domain the integers from 1 to 30), xmin = 0, xmax = 31, xscl = 1, ymin = 0, ymax = 1.5 yscl = 1. Making t-step = 1 will make t take on only the values 1, 2, 3, …, 30 and graph the sequence. The convergence to 1 is apparent:



A sequence graphed in the plane. Notice how the convergence appears.

To graph the points on a number line enter $xt1(t)$ = 1 – (–0.7)^t and $yt1(t)$ = 1. Change the window settings to tmin = 1, tmax = 30, tstep = 1, xmin = 0, xmax = 2, xscl = 1, ymin = 0, ymax = 2, yscl = 1. The points will graph separately on a line. Trace the values to see the convergence.

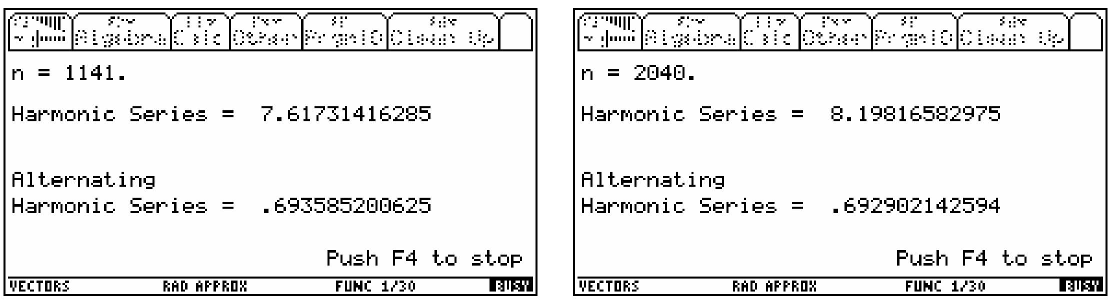


A sequence graphed on a number line. Use the trace feature to see the sequence converge.

Investigation 2: Watching the sequence diverge and converge.

Watching a series converge or diverge can be instructive. The TI89/92+/V200 program "Series" and the TI83/84 programs "Harmonic" and "Series" will let you do this*. Enter a series and the program will give the running total of the sequence of partial sums as the number of terms increases.

The harmonic series and the alternating harmonic series are particularly instructive since they diverge and converge very slowly. Let the program run for a class period. As more terms are added, it will be apparent that the harmonic series continues to get larger. The alternating harmonic series oscillates between larger and smaller values but digit-by-digit stops changing. (Idea contributed by Mark Howell)



The Harmonic and Alternating Harmonic Series

Use the Series program and enter the power series for, say, $\cos\left(\frac{\pi}{3}\right)$ for which $a_n = (-1)^n \dfrac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$ and watch it converge to 0.5.

Investigation 3:

This may be used as soon as the students understand the tangent line approximation idea, and then reused at the beginning of the study of Taylor Series. It tries to "sneak up" on the power series without any mention of the term.

Consider the function $f(x) = \ln(x),\ x > 0$

a. Write the tangent line approximation for f at $x = 2$.

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* These programs may be downloaded from http://www.dsmarketing.com/books_teachcalAB.html

b. Write a quadratic approximation to f at $x = 2$, by finding a polynomial of the form $a+b(x-2)+c(x-2)^2$, Determine the value of *a, b,* and *c* so that the quadratic

   i. Contains the point $(2, \ln(2))$,

   ii. Has the same first derivative as f at $x = 2$, and

   iii. Has the same second derivative as f at $x = 2$.

c. Graph f, the tangent line and the quadratic you found on the same axes. Discuss what you see.

d. Try to find a cubic polynomial of the same form (i.e., with all three derivatives equal to those of f at $x = 2$). Graph it with the others and discuss.

e. Try all this with a different function such as $y = \sin(x)$.

Investigation 4:

Let $f(x) = x^3 + x^2 - 14x - 24$

   i. Write the power series for f centered at $x = 2$.
   ii. Expand the terms of the power series and simplify.
   iii. Is this an accident or will this happen with any polynomial? Explain

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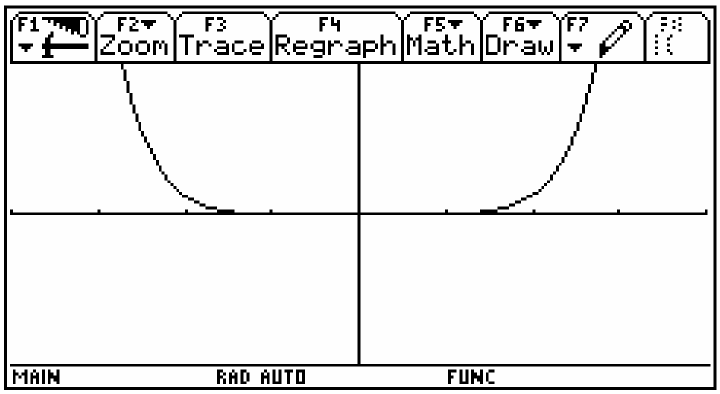
Investigation 5:

1988 BC 4: Determine all values of x for which the series $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$ converges. Justify your answer.

(Answer: $-\frac{1}{2} \le x < \frac{1}{2}$, Using Ratio Test, L'Hôpital's Rule, Comparison Test and Alternating Series Test.)

Investigation 6: Seeing the error in a Taylor Polynomial:

If you know the function and its Taylor Polynomial you may compare values in the interval of convergence to find the error. A graphing calculator makes this easy. Not only can you see the interval of convergence but you can also graph the error function. With the function entered as y1 and the Taylor Polynomial as y2, graph y3 = $|y1(x) - y2(x)|$. Use a window that is very narrow in the vertical direction.



The graph of the error in using the fifth degree Taylor Polynomial to approximate sin(x). The error is very close to zero between –1 and 1. The window is x[–4, 4] by y[–0.2, 0.2].

Investigation 7: 2000 BC 3: The Taylor series about x = 5 for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at x = 5 is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about x = 5.
(b) Find the radius of convergence of the Taylor series for f about x = 5.
(c) Show that the sixth-degree Taylor polynomial for f about x = 5 approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(Answers: (a) $P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$ (b) Radius = 2 (c) Using alternating series test. Note that if f(4), then *not* alternating.)

Investigation 8: 1999 BC 4: The function f has derivatives of all orders for all real numbers *x*. Assume *f*(2) = –3, *f* '(2) = 5, *f* "(2) = 3, and *f* '''(2) = –8.

(a) Write the third-degree Taylor polynomial for *f* about *x* = 2 and use it to approximate *f*(1.5).

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval [1.5; 2].Use the **Lagrange error bound** on the approximation to *f*(1.5) found in part (a) to explain why $f(1.5) \ne -5$.

(c) Write the fourth degree Taylor polynomial, *P*(*x*), for $g(x) = \left(x^2 + 2\right)$ about *x* = 0. Use P to explain why g must have a relative minimum at x = 0.

(Answers: (a) $T_3(f,2) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$; –4.958,

(b) LEB = $\frac{3}{4!}\left|1.5-2\right|^4 = 0.0078125$, $f(1.5) > -4.958\bar{3} - 0.0078125 = -4.966 > -5$

(c) $T_2(f,2)\left(x^2+2\right) = -3 + 5x^2 + \frac{3}{2}x^4$ and from the coefficients

$g'(0) = 0$ and $g''(0) > 0$ therefore a minimum.)

Investigation 9: 1993 BC 5 Let *f* be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of *f*(*x*) about *x* = 0.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about for $x = 0$ for $g(x) = \dfrac{e^{\frac{x}{2}} - 1}{x}$

(c) For the function *g* in part (b), find $g'(2)$ and use it to show that $\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$

(Answers: (a) $e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$

(b) $\frac{e^{x/2} - 1}{x} = \frac{1}{2} + \frac{x}{2^2 2!} + \frac{x^2}{2^3 3!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$ (c) show work

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URLs:

1. The Series and Harmonic Series programs for the TI83+, TH86 and TI89 calculators may be found at www.dsmarketing.com. Click on "New Items" and then on *Teaching AP Calculus*. Scroll to the bottom of the page.

2. *Teaching AP Calculus* by Lin McMullin www.dsmarketing.com . Click on "New Items"

3. Winplot http://math.exeter.edu/rparris/

   Instructions at http://matcmadison.edu/alehnen/winptut/winpltut.htm

4. AP Central Calculus Question of the Month referred to can be found at http://www.linmcmullin.net/QOM_Page.html

AP Exam questions on Power Series:

When graphing calculators were first allowed on the AP Exam there was a change in the style of the questions on power series. The HP48 and later the TI89 calculators are capable of producing symbolic Taylor Polynomials. To avoid giving students with these calculators an advantage, the questions were written in such a way that a CAS would be of no help. The years 1995 through 2001 saw questions that were original, easy to do by hand (if one understood power series), clever, and interesting. Since 2002 the questions appeared on the non-calculator section. Whether this will continue now that there is a no calculator section on the Free-response part of the exam remains to be seen. Look at the pre-1995 and post-1994 questions.

WARNING! The Taylor Approximating Polynomial is *not* the same as the function it approximates. It is important that students writing the AP Exam realize this. See the scoring standards for 1998 BC 3 (a) on which a point is deducted for incorrect use of the "=" sign. Having found T_3, the third degree Taylor Polynomial for a function *f*, they should write $T_3(1.2) \approx f(1.2)$ and not $T_3(1.2) = f(1.2)$.

<u>Traditional style (pre-1995, 2002 – 2006)</u>

1990 BC 5 – a geometric series, integrate, investigate error

1991 BC 5 – a geometric type series, integrate, find interval of convergence

1992 BC 6 – convergence of a series.

1993 BC 5 – find series by substitution

1994 BC 5 – substitution, interval of convergence, error

2002 BC 6 – Find interval of convergence, series for derivative, evaluate at point.

2002 BC (Form B) 6 – Substitute, find interval, analyze related series

2003 BC 6 – Second derivative test using series, approximate, show that series solves Differential equation.

2003 BC (Form B) 6 – Given general derivative, write series, find radius, write series for integral and determine convergence at point.

2004 BC 6 – Write series, Find high-order coefficient, Lagrange error bound, integrate

2004 BC (Form B) 2 – Find derivatives, analyze, Lagrange error bound

2005 BC 6 – Write series, find general coefficient, interval of convergence

2005 (Form B) 3 – Given $f^{(n)}$, max/min 2-DT, write series, find radius of convergence.

2006 BC 5 – Implicit y'', write Taylor Poly, Euler

2006 BC 6 – Interval w/ endpoints, y' and y'' from coefficients max/min

2006 BC (Form B) 6 – Differentiate, integrate, alternating series error bound.

<u>No Calculator style (1995 - 2001)</u>

1995 BC 4 – given values of derivatives, write series, and use it to approximate a value, differentiate.

1996 BC 2 – Given a Maclaurin series, find derivative values (from coefficients), interval of convergence, related power series, recognize familiar function.

1997 BC 2 – Given Taylor Polynomial, find coefficients, derivative and integral

1997 – Multiple-choice: 14 (sum of Geometric series), 17 (recognize series for ln *x*), 20 (determine interval of convergence), 24 (substitute into known series), 76 (convergence tests).

1998 BC 3 – Given derivatives, write power series, differentiate, substitute, integrate (on scoring standard notice the deduction for incorrect use of the equal sign).

1998 – Multiple-choice: 14 (evaluate power series), 18 (convergence tests), 22 (integral test, *p*-series), 27 (differentiate and evaluate Taylor series), 83 (given function and its power series do an error analysis), 84 (interval of convergence with endpoint consideration), 89 (recognize power series and then use *f* to solve equation)

1999 BC 4 – Given derivatives, write Taylor Polynomial, **Lagrange error bound**. Use Taylor polynomial to justify max/min.

2000 BC 3 – Given general expression for derivatives, write Taylor Polynomial, find radius of convergence, error analysis.

2001 BC 6 – Given power series, find interval of convergence, find limit (this limit is actually the derivative of the function!), integrate, find sum of integral series.

Some Problems

The following problems are adapted from *Multiple-choice and Free-response Questions in Preparation for the AP Calculus (BC) Examination*, by David Lederman assisted by Lin McMullin. www.dsmarketing.com You may reproduce these questions for your class.

1. Write the first four nonzero terms in the Maclaurin series for xe^{-x}.

2. $\sum_{k=0}^{\infty}\left(-\frac{\pi}{3}\right)^k =$

(A) $\dfrac{1}{1-\frac{\pi}{3}}$ (B) $\dfrac{\frac{\pi}{3}}{1-\frac{\pi}{3}}$ (C) $\dfrac{3}{3+\pi}$ (D) $\dfrac{\pi}{3+\pi}$ (E) The series does not converge.

3. Let E be the error when the Taylor polynomial $T(x)=x-\dfrac{x^3}{3!}$ is used to approximate $f(x)=\sin(x)$ at $x = 0.5$. Which of the following is true?

(A) $|E|<0.0001$ (B) $0.0001<|E|<0.0003$ (C) $0.0003<|E|<0.005$

(D) $0.005<|E|<0.007$ (E) $0.007<|E|$

4. The Taylor series of a function $f(x)$ about $x = 3$ is given by

$$f(x)=1+3(x-3)+\frac{5(x-3)^2}{2!}+\frac{7(x-3)^3}{3!}+\cdots+\frac{(2n+1)(x-3)^n}{n!}$$

What is the value of $f'''(3)$ and $f^{(7)}(3)$?

5. What are all values of x for which the series $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$ converges?

(A) $-1\le x\le 1$ (B) $-1\le x<1$ (C) $-1<x\le 1$

(D) $-1<x<1$ (E) All real numbers x.

6. $\sum_{k=1}^{n} \frac{(-1)^n (\pi)^{2n}}{(2n)!} =$

7. Let $f(x)$ be the function defined by the power series $f(x) = \sum_{k=0}^{\infty} 2x^n$. If $g'(x) = f(x)$ and $g(0) = 2$ then $g(x) =$

8. Let $f(x)$ be a function with the following properties:

(i) $f(0) = -2$

(ii) $f'(x) = 3f(x)$

(iii) The n^{th} derivative of f, $f^{(n)}(x) = 3f^{(n-1)}(x)$

(a) Give the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Find $f(x)$ by solving the differential equation in (ii) with the initial condition in (i).

(c) Graph and label both f and the third degree Maclaurin polynomial of f on the axes below and label each. [Window is [–2,2] by [–40, 10])

Answers:

1. $x - x^2 + \frac{1}{2!}x^3 - \frac{1}{4!}x^5 + - \cdots$;
2. (E);
3. B;
4. 7, 15;
5. C;
6. $\cos(\pi) = -1$

7. $2 + \sum_{k=1}^{\infty} \frac{2x^n}{n}$ the 2 is the constant of integration .

8. (a) $-2 - 6x - 9x^2 - 9x^3 - \cdots - \frac{2 \cdot 3^n \cdot x^n}{n!}$,
(b) $f(x) = -2e^{3x}$, (c) at right