Going Up?

By Lin McMullin

One of the most frequently asked questions on the AP Calculus Electronic Discussion Group is, "If a function is increasing or decreasing on an interval should the endpoints be included?" And with it a similar question, "Is a function like $y = x^3$ increasing at the origin where the derivative is zero?"

I think the confusion results from too much calculus and not enough basic mathematics.

Several years before studying calculus students learn about functions and their properties. They also learn about definitions and theorems. The concept of increasing and decreasing functions comes up naturally when the properties of graphs are discussed. The definition of these two terms should be presented then, both because they are important to the topic and because they are good definitions to use to teach about definitions. And knowing about definitions is important in all areas of mathematics.

Here is the definition of increasing (from here on I'll write about increasing only) followed by some notes on how to "unpack" it for a high school class algebra class.

$$\underbrace{\text{A function } f \text{ is increasing on an interval } I}_{2}\ \underbrace{\text{if, and only if,}}_{1}$$
$$\underbrace{\text{for all } x_1 \text{ and } x_2 \text{ in } I,}_{3}\ \underbrace{\text{if } x_1 < x_2 \text{ then } f(x_1) < f(x_2)}_{4}$$

Some points to emphasize:

- Part 1: *All definitions are <u>if, and only if,</u> statements* (also called biconditional statements). Sometimes the phrase does not appear (usually for rhetorical reasons) but it is implied. It means that what's written before the phrase and what's written after it are equivalent. If one part is given or shown to be true, then the other part must also be true. If one part is given or shown to be false, then the other part must also be false.

- Part 2: The term being defined is not *increasing* it is *increasing on an interval*. That's important; there is always an interval involved. Since one point does not an interval make, you cannot say a function is increasing *at a point*. That's unfortunate because many people, books, and test do just that.
- Part 3: In mathematical contexts the words *any*, *every*, and *all* are interchangeable and should be interchanged. "For all x_1 and x_2 in *I*" means "for *any* two values x_1 and x_2 in the interval" and "for *every* pair of values x_1 and x_2 in the interval" whatever follows will have to work. For this definition you do not get to choose which two points to work with; you have to consider all pairs (any pair, every pair). It also means that if just one pair does not work you are out of luck.
- Part 4: The *if, …, then* part is the test that the pairs will have to pass. The condition that $x_1 < x_2$, tell us to distinguish the larger and smaller value of *x*; the one to the left from the one to the right. If it is always true that the function value of the left point $f(x_1)$ is less than the function value of the right point $f(x_2)$, then the function is increasing. So the definition requires that every point must have a greater function value than all (any, every) points to its left and lesser function value than all (any, every) points to its right.

We can now answer the first question – and without any calculus. If the endpoints are in the domain of the function, then they must be included in intervals where the function is increasing or decreasing. For example, $\sin(x)$ is increasing on the closed interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; $\sin\left(-\frac{\pi}{2}\right)$ is less than the sine of all the other numbers in this interval. Likewise, $\sin\left(\frac{\pi}{2}\right)$ is greater than all the other sine of all the other numbers in the interval, By similar reasoning, $\sin(x)$ decreases on the closed interval $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.

The fact that $x=\frac{\pi}{2}$ is in both intervals doesn't matter; functions don't increase or decrease at a single point. When climbing a hill you climb all the way from the bottom to the top; then when you go down the hill your altitude decreases all the way from the top to the bottom.

A student may ask, "But Mr. McMullin, $\sin\left(-\frac{\pi}{2}\right) < \sin\left(\frac{3\pi}{4}\right)$, how come the function isn't increasing all the way to $\frac{3\pi}{4}$?" (Good students will do that!) The answer is that there are other pairs of points in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{4}\right]$, such as $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$, for which the value on the right is less than that on the left. You must consider *every* (*all*, *any*) pairs.

Now to the calculus: There is a theorem in calculus that says something like, "If the derivative of a function is positive on an interval then the function is increasing on the interval."

Nice theorem and useful (but it could be stated better). It gives you a quick way of finding where a function is increasing: find where the derivative is positive. But the theorem does not purport to tell you what's happening where the derivative is zero or does not exist. Also, the converse is false; functions may increase on intervals where their derivative is zero or undefined. So the theorem gets you started but you must consider the endpoints and points where the derivative is zero or undefined separately before giving the complete interval.

This answers our second question, $y = x^3$ is increasing on any interval including those containing the origin where the derivative is zero. The theorem above does not apply at the origin, so go back to the definition. All the points have *y*-values larger than those to their left: the function is increasing everywhere.

The correct statement of the theorem above is this (notice the open and closed intervals):

> If f is a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and if $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

Finally, a very common question in textbooks and on tests is this: "Is the function f increasing at the point where $x = a$?" Since the question is so ubiquitous, it is probably not a good idea to say, "No, functions cannot increase at a point only on an interval."

While there is no consensus on this, it is reasonable to say that a function is increasing at a point if there is an open interval containing the point on which the function is increasing. In other words, if a function is increasing on an open interval it is increasing at each point of the interval. Furthermore, if a function is twice differentiable at $x = a$ and if $f'(a) > 0$ then there is an open interval containing $x = a$ on which the function is increasing. (This is because the derivative is a two-sided limit, and along with the existence of the second derivative which implies the continuity of the first derivative, implies the existence of the open interval.)