NOTES

Quartic Polynomials and the Golden Ratio

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Suppose we have a pentagram in the xy-plane, oriented as in FIGURE 1a, and want to find a quartic polynomial whose graph passes through the three vertices indicated. Out of infinitely many possibilities, there is exactly one quartic polynomial that attains its minimum value at both of the two lower vertices. This graph—shaped like a smooth W with its local maximum at the upper vertex—is shown in FIGURE 1b. Now, how does the graph continue? Will it touch the pentagram again on its way up to infinity? As it turns out, the graph passes through two more vertices, as shown in FIGURE 1c. Furthermore, the two points where the graph crosses the interior of a pentagram edge lie exactly below two other vertices, as shown in FIGURE 1d.

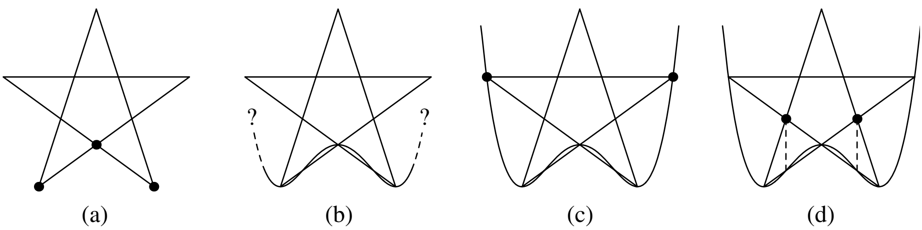


Figure 1 A pentagram and a quartic polynomial

Knowing that length ratios within the pentagram are determined by the golden ratio, we realize that this quartic polynomial has some regularities governed by the same ratio. As we will see, there are many more such regularities. Furthermore, they apply to *all* quartic polynomials with inflection points. This will be clear once we realize that the different quartics are all related by an *affine transformation*.

Symmetric quartic We investigate graphs of quartic polynomials with inflection points by means of certain naturally defined points and length ratios. As an example, we consider the function $f(x) = x^4 - 2x^2$, shown in FIGURE 2. (This quartic's shape differs slightly from the one in FIGURE 1 and is chosen to simplify calculations.) We define $P_0(x_0, y_0)$ as the point where the third derivative vanishes, so that $f'''(x_0) = 0$. The tangent points of the double tangent (the unique line that is tangent to the graph at two points) are called P_1 and P_2. The points where the tangent at P_0 intersects the graph are P_3 and P_4. We number points so that those to the left of P_0 have odd index, while those to the right have even index.

The line through P_0 and P_1 intersects the graph in two additional points, called P_6 and P_7. Similarly, the line through P_0 and P_2 has the additional intersection points P_5

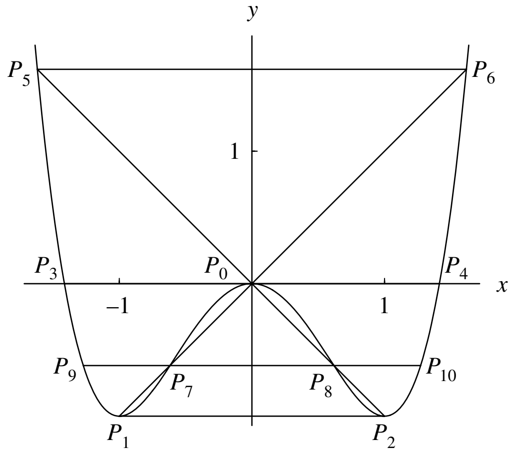


Figure 2 The quartic $x^4 - 2x^2$

and P_8. (The inflection points might appear to be P_7 and P_8, but this is not so.) The line through P_7 and P_8 intersects the graph in P_9 and P_{10}.

For this graph, we easily find the coordinates $P_0(0, 0)$, $P_2(1, -1)$, and $P_4(\sqrt{2}, 0)$. Symmetry guarantees that the points P_1 through P_{10} are located symmetrically about the y-axis. The coordinates of P_5 through P_{10} turn out to involve the golden ratio, $\varphi = (1 + \sqrt{5})/2$. Using the relations $\varphi^2 = \varphi + 1$ and $\varphi^{-2} = 1 - \varphi^{-1}$, we calculate three function values: $f(\varphi) = \varphi$ and $f(\varphi^{-1}) = f(\sqrt{\varphi}) = -\varphi^{-1}$. From these calculations and the fact that the points P_5 through P_8 lie on the lines $y = \pm x$, we find the coordinates $P_6(\varphi, \varphi)$, $P_8(1/\varphi, -1/\varphi)$, and $P_{10}(\sqrt{\varphi}, -1/\varphi)$. From these coordinates, the following relations between line segment lengths follow quickly:

$$P_3P_4 = \sqrt{2}P_1P_2, \quad P_5P_6 = \varphi P_1P_2, \quad P_7P_8 = P_1P_2/\varphi, \quad P_9P_{10} = \sqrt{\varphi}P_1P_2. \tag{1}$$

Our next step is to show that these relations carry over to the general case.

General quartic We will *not* proceed by deriving general expressions for the coordinates of the points P_0 through P_{10}. Instead, we shall see that the graph of every quartic polynomial with inflection points can be obtained as the image of the graph of the symmetric quartic above subject to an appropriate affine transformation. An affine transformation consists of an invertible linear transformation followed by translation along a constant vector. An affine transformation of the plane has the following properties: Straight lines are mapped to straight lines, parallel lines to parallel lines, and tangents to tangents, while length ratios between parallel line segments are preserved [**1**, chapter 2].

Consider the symmetric quartic

$$f(x) = x^4 + wx^2, \quad w < 0, \tag{2}$$

and a general quartic with inflection points,

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0.$$

Define x_0 by $g'''(x_0) = 0$ (so $x_0 = -b/4a$) and $k = \sqrt{g''(x_0)/2aw}$. (The existence of two inflection points implies that $g''(x_0)$ and a have opposite signs, and since w is negative, k is real.) The map $(x, y) \mapsto (\bar{x}, \bar{y})$ given by

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g'(x_0) & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & ak^4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ g(x_0) \end{pmatrix} \tag{3}$$

is an affine transformation consisting of a scaling transformation (with a scaling factor k in the x-direction and a scaling factor $|a|k^4$ in the y-direction), a reflection about the x-axis if a is negative, a shear in the y-direction, and a translation. In components, we have the equations

$$\bar{x} = kx + x_0, \quad \bar{y} = ak^4 y + g'(x_0)kx + g(x_0).$$

Suppose (x, y) lies on the graph of f, that is, $y = f(x)$. Substituting $y = x^4 + wx^2$ and $x = (\bar{x} - x_0)/k$ into the expression for $\bar{y}$ yields

$$\bar{y} = a(\bar{x} - x_0)^4 + \frac{1}{2} g''(x_0)(\bar{x} - x_0)^2 + g'(x_0)(\bar{x} - x_0) + g(x_0).$$

The right-hand side is the fourth-degree Taylor polynomial of g at x_0 and is therefore identical to $g(\bar{x})$. Thus, $(\bar{x}, \bar{y})$ lies on the graph of g, so the above transformation indeed maps the graph of f to the graph of g.

Moreover, the origin, where $f'''(x) = 0$, is mapped to $(x_0, g(x_0))$, where $g'''(x) = 0$. From this and the general properties of affine maps it follows that each of the points P_0 through P_{10} on the graph of f is mapped to the analogously defined point on the graph of g. (We will use the same notation P_i for points on both graphs.) The results for the case $w = -2$ then show that the line segments P_1P_2, P_3P_4, ... , P_9P_{10} on the graph of g are all parallel, are bisected by the vertical line through P_0, and satisfy the relations (1). FIGURE 3 illustrates this for the quartic $2x^4 - x^3 - 2x^2 + x + 1$. Note that P_0 divides the line segments P_6P_1 and P_5P_2 according to the golden ratio, P_7 divides P_0P_1 according to the golden ratio, and analogously for P_8 and P_0P_2.

Further characteristic ratios We now define some more points on the graph of a quartic, starting with the point P_0 from FIGURE 3 and the inflection points P_{11} and P_{12};

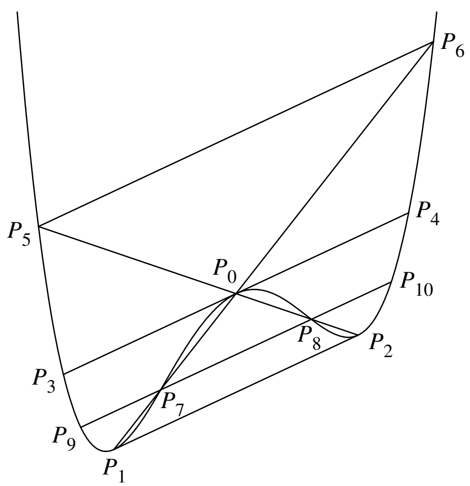


Figure 3 The quartic $2x^4 - x^3 - 2x^2 + x + 1$ and the points P_0 through P_{10} (axes not shown)

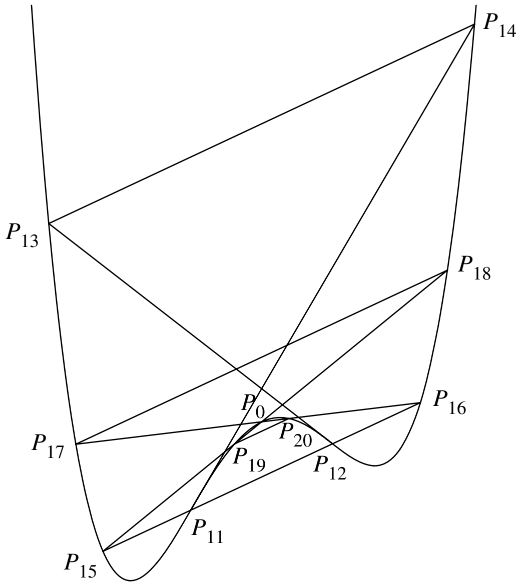


Figure 4 The quartic $2x^4 - x^3 - 2x^2 + x + 1$ and the points P_{11} through P_{20}

see FIGURE 4. The tangents at the inflection points intersect the graph at the points P_{13} and P_{14}. The line through the inflection points intersects the graph at P_{15} and P_{16}. The line through P_0 and P_{15} intersects the graph at P_{18} and P_{19}, while the line through P_0 and P_{16} intersects the graph at P_{17} and P_{20}. The list of statements may now be extended:

THEOREM. *Let $P_0, \dots, P_{20}$ be points defined as above on the graph of a quartic polynomial with inflection points, and $\varphi = (\sqrt{5}+1)/2$. Then:*

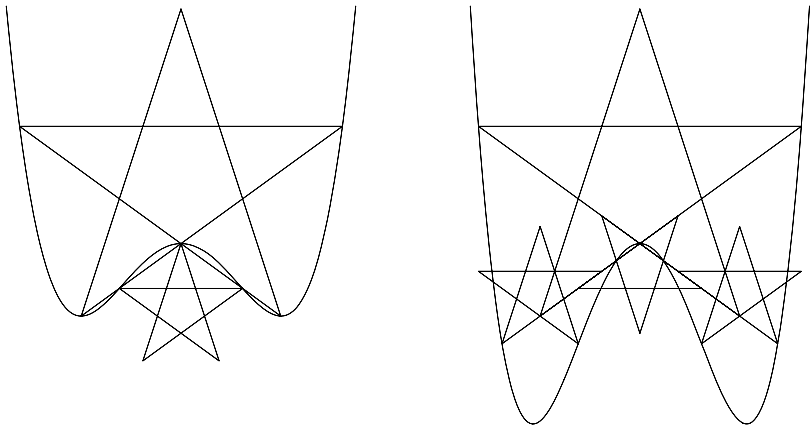
1. *The line segments $P_{2n-1}P_{2n}$ $(n = 1, \dots, 10)$ are all parallel.*
2. *Intersection points of the graph and a line parallel to the tangent in $P_0(x_0, y_0)$ are symmetrically located about the point on the line with $x = x_0$.*
3. $P_3P_4 = \sqrt{2}P_1P_2$
4. $P_5P_6 = \varphi P_1P_2$
5. $P_7P_8 = P_1P_2/\varphi$
6. $P_9P_{10} = \sqrt{\varphi}P_1P_2$
7. $P_{11}P_{12} = P_1P_2/\sqrt{3}$
8. $P_{13}P_{14} = 3P_{11}P_{12}$
9. $P_{17}P_{18} = \varphi^2 P_{11}P_{12}$
10. $P_{15}P_{11} = P_{12}P_{16} = P_{11}P_{12}/\varphi$
11. $P_{19}P_{20} = P_{11}P_{12}/\varphi^2$

Proof. The new statements (2, 7–11, and part of 1) may be verified relatively easily for the quartic $x^4 - 6x^2$, that is, $f(x)$ from (2) with $w = -6$. (Since then $f''(\pm 1) = 0$, this quartic is simpler to use for P_{11} through P_{20} than $x^4 - 2x^2$.) Now, inflection points are mapped to inflection points by an affine map. (Indeed, $g''(\bar{x}) = ak^2 f''(x)$ in our case.) Then, the arguments made earlier apply here as well. ■

The properties of the line through the inflection points P_{11} and P_{12} (statement 10, and in part 1) have been pointed out earlier [**2**], as has the symmetry property (statement 2) and the fact that P_0 is the point where the tangent is parallel to the double tangent (a consequence of statement 1) [**2, 3**]. I have found no reference to the other relations, including, in particular, the five occurrences of the golden ratio.

Our affine transformation (3) shows that the graph of a general quartic function may be regarded as an originally symmetric graph that has been sheared in the y-direction and moved. Considering this, the properties regarding parallelism and symmetry (for instance, that the line segments $P_{15}P_{11}$ and $P_{12}P_{16}$ have equal length) become obvious. The same applies to ratios between areas, since a scaling transformation changes all areas by a constant factor, while a shear preserves areas. For example, it is known that the line through the inflection points of an arbitrary quartic function cuts off three areas that are in the ratio of $1 : 2 : 1$. This is readily verified for $f(x)$ from (2) with $w = -6$ by checking that $\int_0^{\sqrt{5}} (f(x) - (-5))\, dx = 0$, whereby it is immediately proven generally.

Quartic polynomials and pentagrams Returning to FIGURE 1, we see that it is just an example of statements 4 and 5 of the theorem. The same can be illustrated by FIGURE 5a, where the smaller pentagram fits exactly into the inner pentagon of the larger pentagram (meaning the linear size ratio is $1 : \varphi^2$). Similarly, as the reader may check, FIGURE 5b illustrates statements 9, 10, and 11. In each of these graphs, three points are given, two of which are specified as minimum points, (a), or inflection points, (b). This completely determines the graphs; they will automatically pass through four or six more vertices. The possibility of finding such simple constellations of pentagrams and quartic graphs reflects the occurrence of the golden ratio in quartic polynomials.



(a) (b)

Figure 5 Quartic polynomials passing through pentagram vertices

To summarize, we have found simple characteristic length ratios on the graph of a quartic polynomial with inflection points, including several occurrences of the golden ratio. These length ratios are left invariant by an affine transformation that relates a symmetric quartic to a general quartic with inflection points.

REFERENCES

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3. F. Irwin and H. N. Wright, Some properties of polynomial curves, *Annals Math.* (2nd Ser.) **19** (1917) 152–158.