PHI on Cubics! phi!

By Lin McMullin

The Golden Ratio appears again; this time in cubic polynomials!

This result was sent to me by David Tschappat a senior math major at Our Lady of the Lake University, San Antonio, TX. It is, I believe, an original discovery by him.

The result is presented here as an exercise with my solution.

Consider a cubic polynomial, $f(x)$, with a point of inflection at $x = r$ and (local) extreme values at $x = p_1$ and $x = p_2$. Consider two other points $x = q_1$ and $x = q_2$ such that

$$q_1 < p_1 < r < p_2 < q_2$$

And such that the numbers are equally spaced, h units apart; i.e.,

$$q_2 - p_2 = p_2 - r = r - p_1 = p_1 - q_1 = h$$

(Note: A cubic polynomial is symmetric about its point of inflection. This implies that $p_2 - r = r - p_1$. Then q_1 and q_2 are chosen so the five points will be equally spaced, h units apart.)

1. Warm up exercise 1: Show that $f(q_1) = f(p_2)$ and $f(q_2) = f(p_1)$
2. Warm up exercise 2: Show that $f'(q_1) = f'(q_2)$
3. The real exercise: Recall that the Golden ratio $\Phi = \frac{1}{2}\left(1+\sqrt{5}\right)$ and its conjugate $\phi = \frac{1}{2}\left(1-\sqrt{5}\right)$. Show that the line through the local extreme point $\left(p_2, f(p_2)\right)$ that is parallel to the tangent line to f at $\left(q_1, f(q_1)\right)$, intersects the cubic at

$$x_1 = p_2,$$
$$x_2 = \Phi \cdot p_2 + \phi \cdot q_1, \text{ and}$$
$$x_3 = \Phi \cdot q_1 + \phi \cdot p_2$$

Solution

1. Since the graphs of cubic polynomials are symmetric to their point of inflection, we can, without loss of generality, translate *f* so that its point of inflection is at the origin. Let *g* be the image of *f*. Then,

> The image of $q_1 = -2h$,
> The image of $p_1 = -h$, the x-coordinate of the extreme value,
> The image of $r = 0$, the x-coordinate of the point of inflection,
> The image of $p_2 = h$, the x-coordinate of the extreme value,
> The image of $q_2 = 2h$, and
> The image of $f(x)$ is $g(x) = kx^3 - 3h^2kx$, where k is a non-zero constant

2. First warm up exercise:

$$g(-2h) = -8h^3k + 6h^3k = -2h^3k$$
$$g(h) = kh^3 - 3h^3k = -2h^3k$$
$$\therefore g(-2h) = g(h)$$

And likewise $g(-h) = g(2h) = 2h^3k$. This means that the pairs of points are located horizontally from each other.

3. Second warm up exercise:

$$g'(x) = 3kx^2 - 3h^2k$$
$$g'(-2h) = 3k(-2h)^2 - 3h^2k = 9h^2k$$
$$g'(2h) = 3k(2h)^2 - 3h^2k = 9h^2k$$
$$\therefore g'(-2h) = g'(2h)$$

This should be obvious since cubic polynomials are symmetric to their point of inflection and also from the fact that $g'(x)$ is an even function.

4. The real exercise: The line through $(h, g(h))$ with slope of $g'(-2h) = 9h^2k$ is

$$\begin{aligned} y(x) &= g(h) + g'(-2h)(x-h) \\ &= -2h^3k + 9h^2k(x-h) \\ &= 9h^2kx - 11h^3k \end{aligned}$$

Then to find the points of intersection solve

$$g(x) = y(x)$$
$$kx^3 - 3h^2kx = 9h^2kx - 11h^3k\ .$$
$$x^3 - 12h^2x + 11h^3 = 0$$

Since we know one solution is $x = h$ this expression may be factored by hand by using synthetic division to find the depressed equation, a quadratic, which may then be factored by using the quadratic formula, as shown below. (I of course used a CAS).

$$(x-h)\left(x^2 - hx - 11h^2\right) = 0$$
$$(x-h)\left(x + \tfrac{1}{2}\left(1-3\sqrt{5}\right)h\right)\left(x + \tfrac{1}{2}\left(1+3\sqrt{5}\right)h\right) = 0$$

$$x_1 = h, \qquad x_2 = \frac{\left(-1+3\sqrt{5}\right)}{2}h, \qquad x_3 = \frac{\left(-1-3\sqrt{5}\right)}{2}h$$

Finally,

$$\Phi \cdot h + \phi \cdot (-2h) \quad = \quad \frac{1+\sqrt{5}}{2}h + \frac{1-\sqrt{5}}{2}(-2h) \quad = \quad \frac{\left(-1+3\sqrt{5}\right)}{2}h \quad = \quad x_2$$
$$\Phi \cdot (-2h) + \phi \cdot h \quad = \quad \frac{1+\sqrt{5}}{2}(-2h) + \frac{1-\sqrt{5}}{2}h \quad = \quad \frac{\left(-1-3\sqrt{5}\right)}{2}h \quad = \quad x_3$$

QED

By symmetry, a similar result exists for the line through the other extreme point, $\left(-h, g(-h)\right)$, with a slope of $g'(2h)$. The intersection points are at $x_1 = -h$ and

$$x_2 = \Phi \cdot (-h) + \phi \cdot (2h) \quad = \quad \frac{1+\sqrt{5}}{2}(-h) + \frac{1-\sqrt{5}}{2}(2h) \quad = \quad \frac{\left(1-3\sqrt{5}\right)}{2}h,$$
$$x_3 = \Phi \cdot (2h) + \phi \cdot (-h) \quad = \quad \frac{1+\sqrt{5}}{2}(2h) + \frac{1-\sqrt{5}}{2}(-h) \quad = \quad \frac{\left(1+3\sqrt{5}\right)}{2}h$$

– End –