Closest Point Problem Solutions

Original exercise using the point $(0,a)$:

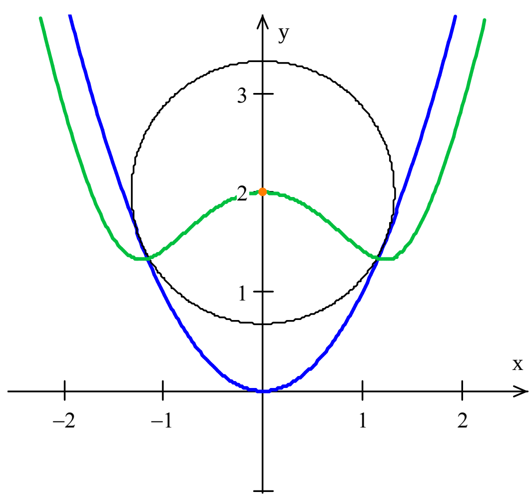
$$z(x)=\sqrt{x^2+\left(x^2-a\right)^2}$$

$$z'(x)=\frac{2x+2\left(x^2-a\right)(2x)}{2z(x)}=\frac{x\left(2x^2-2a+1\right)}{z(x)}$$

$$z'(x)=0 \text{ when } x=0, x=-\sqrt{a-\tfrac{1}{2}}, x=\sqrt{a-\tfrac{1}{2}}$$

$$\text{When } a=2, x=0, x=-\sqrt{\tfrac{3}{2}}, x=\sqrt{\tfrac{3}{2}}$$

Exercise 1: The local maximum distance to the origin is of course 2, so the graph of $z(x)$ goes through the center of the circle. The minimums are directly below the place where the circle and parabola intersect. Their distance above the x-axis is the same as distance from the closest point to (0, 2). .

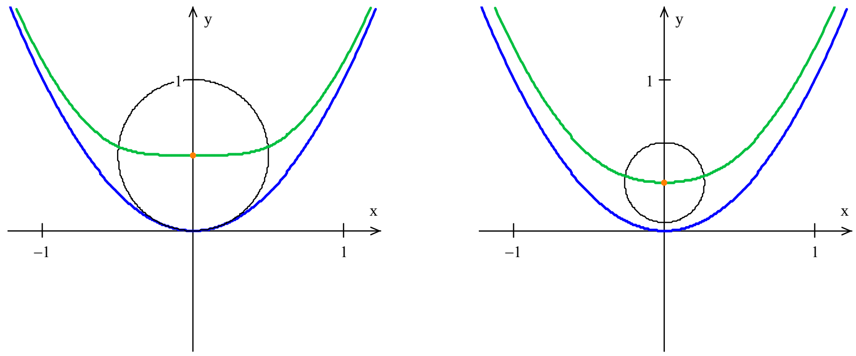


Exercise 2: The closest points occur when $x=\pm\sqrt{a-\tfrac{1}{2}}$. The closest points are $\left(\pm\sqrt{a-\tfrac{1}{2}}, a-\tfrac{1}{2}\right)$, Why?

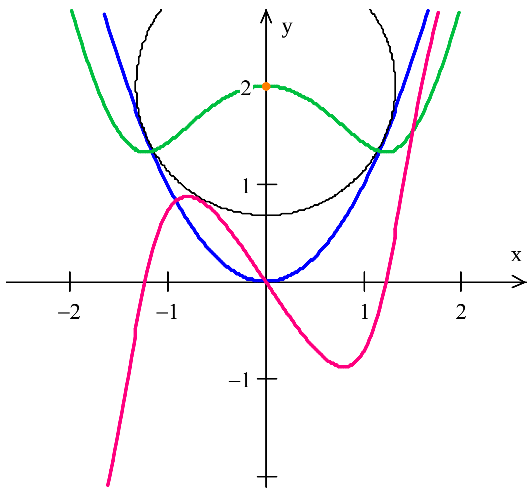
The radius of the circle (the distance from the closest point to the center) touching the closest point is $\sqrt{a-\tfrac{1}{4}}$. When a = ½, x = 0. Here the three critical points are coincident, see left figure below. The origin is the closest, in fact the only point where the circle touches the parabola.

For $\tfrac{1}{4}<a<\tfrac{1}{2}$ the radius of the circle is too short to reach the origin, see the center figure below (a = 0.32).

For values of $a \le \frac{1}{4}$ there is no circle. The closest point is still the origin, but the circle idea breaks down.



Exercise 3 and 4: The zeros of the derivative should be directly below the point where the circle and parabola are tangent and below the low points on the graph of $z(x)$. Due to the symmetry of the parabola, $z(x)$ is an even function and $z'(x)$ is an odd function. (Note: the graph of $z(x)$ does not intersect the parabola at the point of tangency.)



Project: The Winplot equations are:

Parabola:	y = x^2
Circle:	x^2+(y-a)^2=((a-.5)+((a-.5)-a)^2)
Center of Circle:	(x,y) = (0,a)
Distance z(x)	y = sqrt(x^2+(x^2-a)^2)
Derivative z'(x)	y = (x+(x^2-a)(2x))/sqrt(x^2+(x^2-a)^2)

Exercise 5: The coordinates of the points of inflection of the derivative are the zeros of the third derivative of $z(x)$, using a TI-Nspire:

Define $z(x)=\sqrt{x^2+(x^2-a)^2}$	*Done*
$\text{solve}\left(\frac{d^3}{dx^3}(z(x))=0,x\right)$	$x=-\sqrt{a}$ or $x=\sqrt{a}$ or $x=0$ or $a=\frac{1}{4}$

This means the left and right POIs of the derivative have the same *x*-coordinates as the point on the parabola directly left and right of the center of the circle. $\left(-\sqrt{a},\left(-\sqrt{a}\right)^2\right)=\left(-\sqrt{a},a\right)$ and $\left(\sqrt{a},\left(\sqrt{a}\right)^2\right)=\left(\sqrt{a},a\right)$. The origin is also a POI. At a = ¼ , $z'''(x)=0$.