Notes about *Speed* for AP Calculus Teachers

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The current AP* Calculus Course Description under Applications of the Derivative includes this bullet point: [the] interpretation of the derivative as a rate of change in varied contexts, including, velocity, speed and acceleration." Students should understand that if the position of a moving object is given by a function $s(t)$, then its velocity is given by the position function's derivative, $s'(t)=v(t)$, and its acceleration is given by the position's second derivative, the velocity's first derivative $s''(t)=v'(t)=a(t)$.

Speed is the absolute value of velocity; speed = $|v(t)|$. This is the definition of speed, but hardly enough to be sure students know about speed and its relationship to velocity and acceleration.

Velocity is a vector quantity; that is, it had both a direction and a magnitude. The magnitude of velocity vector is the speed. Speed is a non-negative number and has no direction associated with it. If the object is moving on one dimension, then the sign (+ or –) determines the direction the object is moving. On the AB Calculus exams particles often move on a number line, *i.e.* in one dimension, with right or up being the positive direction and left or down being the negative direction.

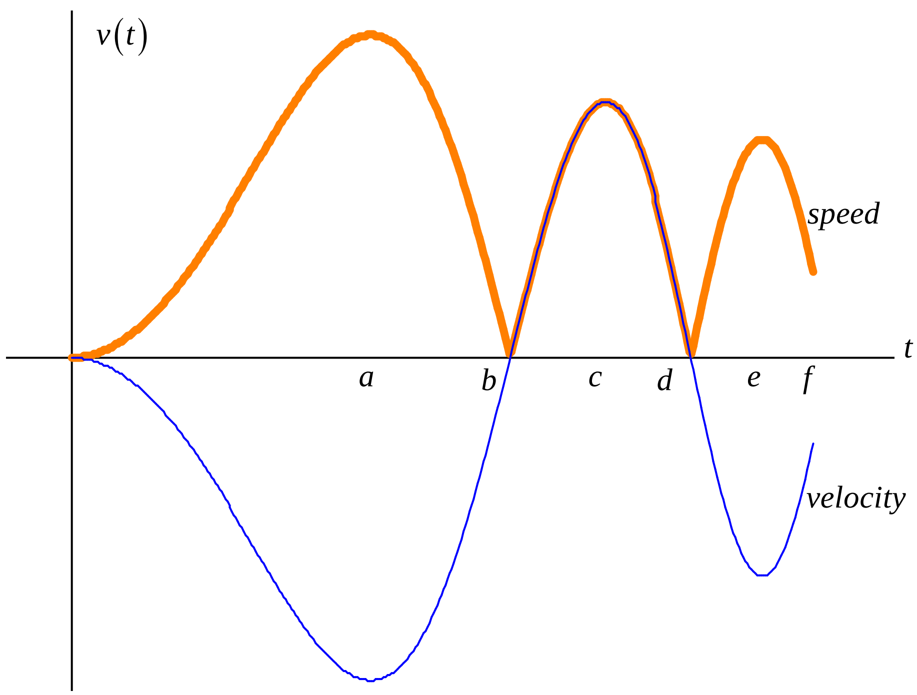
On the BC exam objects move in two-dimensions, i.e. on a plane, in the direction of the sum of its *x*- and *y*-component velocity vectors; the length of velocity vector is its speed. This length is given by $speed=\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}$. A similar relationship applies to three and more dimensions. We will continue here discussing only the one-dimensional case.

Motion questions on the AB exams may have the velocity or position given by an equation, or a graph, or in a table. Calculating the speed is done by finding the velocity at the point required and then finding its absolute value.

The question that seems to trouble students the most is to determine whether the speed is increasing or decreasing. The short answer is that the *speed is increasing when the velocity and* *acceleration have the same sign, and decreasing when they have different signs*. Students may also be asked the speed is greatest or least. Here are two graphical approaches to these concepts.

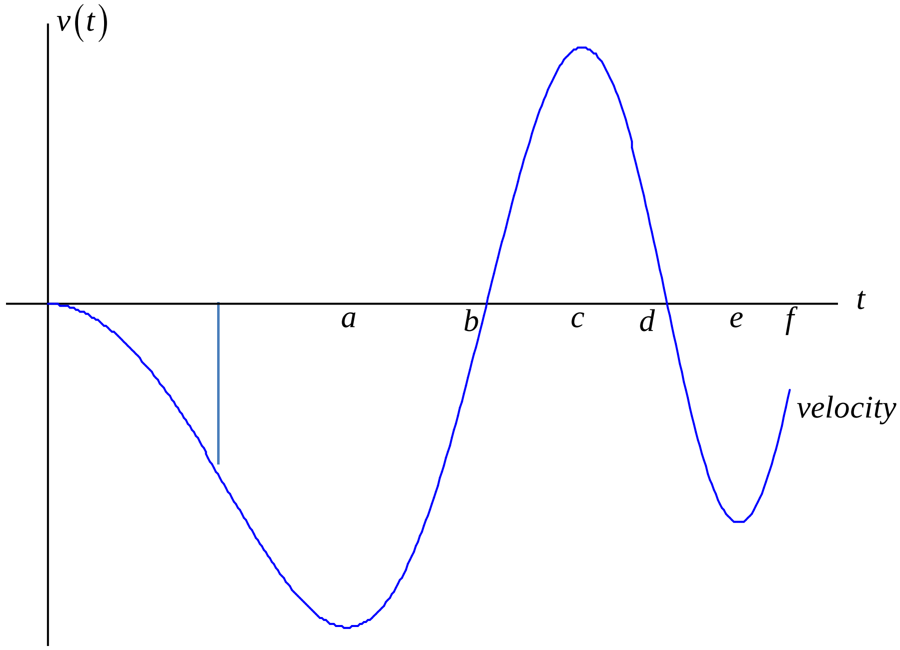
Graphical Approach 1:

The figure below shows the graph of the velocity $v(t)$ (thin graph) of a particle moving on the interval $0 \le t \le f$. The thick graph is $|v(t)|$ the speed; the section where $v(t) < 0$ is reflected over the *x*-axis. (The graphs overlap on [*b*, d].) It is quite east to see when the speed is increasing: [0,*a*], [*b*, *c*] and [*d*,*e*]. Fill in the table below the graph. Recall that the acceleration is the slope of the velocity graph.



Interval	**Velocity** Positive or Negative	**Acceleration** Positive or Negative	**Speed** Increasing or decreasing
[0, *a*]			
[*a*, *b*]			
[*b*, *c*]			
[*c*, *d*]			
[*d*, *e*]			
[*e*, *f*]			

Another way of approaching the concept is this: the speed is the non-directed length of the vertical segment from the velocity's graph to the *t*-axis. Picture the segment shown moving across the graph. When it is getting longer (either above or below the *t*-axis) the speed is increasing. Use this idea to complete the table below the graph.



Interval	**Velocity** Positive or Negative	**Acceleration** Positive or Negative	**Speed** Increasing or decreasing
[0, *a*]			
[*a*, *b*]			
[*b*, *c*]			
[*c*, *d*]			
[*d*, *e*]			
[*e*, *f*]			

Of course this table is the same as the one above.

Thinking of the speed as the non-directed distance from the velocity to the axis makes answering the two questions below easy:

1. What are the values of *t* at which the speed obtains its the local and absolute maximum(s)?
2. When do the minimum speeds occur? What are they?

Students often benefit from a verbal explanation of all this. Picture a car moving along a road going forwards (in the positive direction) its velocity is positive.

- If you step on the gas your acceleration pulls you in the direction you are moving and you speed increases. ($v > 0$, $a > 0$, speed increases)
- Going too fast is not good, so you put on your breaks, you now accelerate in the opposite direction (decelerate?), but you are still moving forward, but slower. ($v > 0$, $a < 0$, speed decreases)
- Finally you stop. Then you shift into reverse and start moving backwards (negative velocity) and you push on the gas to accelerate in the negative direction, so your speed increases. ($v < 0$, $a < 0$, speed increases)
- Then you put on the breaks (accelerate in the positive direction) and your speed decreases again. ($v < 0$, $a > 0$, speed decreases)

Answers

Interval	**Velocity** Positive or Negative	**Acceleration** Positive or Negative	**Speed** Increasing or decreasing
[0, *a*]	Negative	Negative	Increasing
[*a*, *b*]	Negative	Positive	Decreasing
[*b*, *c*]	Positive	Positive	Increasing
[*c*, *d*]	Positive	Negative	Decreasing
[*d*, *e*]	Negative	Negative	Increasing
[*e*, *f*]	Negative	Positive	Decreasing

1. Local maximums at $t = c$ and $t = e$; absolute maximum at $t = a$.
2. Absolute minimums of 0 at $t = 0$, $t = b$, and $t = d$. (Endpoint minimum at $t = f$).